CONTRIBUTIONS TO THE THEORY OF COMBINATORIAL GAMES

by

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Abstract

In this thesis we attempt to address several gaps in the theory of Conway games. In chapters 2 and 3 we introduce several classes of structures, each intended to generalise the class of partisan games found in *On Numbers and Games* [7] and *Winning Ways* [4]. In particular we demonstrate that the "value" of a game [7, p.76] arises as a special case of an adjunction between two categories. The theory of combinatorial game categories [6] is complimented by these structures.

In chapter 4 we introduce a two-sided theory of sets which acts as a foundation for the theory of wellfounded Conway games; this theory is shown to be synonymous with ordinary ZF in the sense of Visser [34].

In chapter 5 we construct, within a nonstandard model of some fragment of ZF, a model of a weak set theory. Parallels are drawn between this construction and the models of topological set theory developed by Forti et al. [14, 12, 15, 13]. We conjecture ways in which this construction may be augmented. Finally, we discuss how this construction can be generalised to the study of combinatorial games and the structures discussed above.

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CHAPTER 1

INTRODUCTION

1.1 Conway games

In the 1970s and '80s Berlekamp, Conway and Guy developed a theory of combinatorial games. These games, hereafter referred to as *Conway games*, are remarkable for their rich structure; see *On Numbers and Games* (we adopt the usual abbreviation, ONAG) [7], *Winning Ways for Your Mathematical Plays* (which we shorten to Winning Ways) [4], or for a gentler but clearer introduction, Schleicher and Stoll [33].

Conway games may be viewed as two-sided containers, the elements of each side being Conway games themselves. There are, therefore, parallels with pure set theory. Indeed, in ONAG Conway discusses the creation or "birth" of games in a way analogous to the creation of sets in the von Neumann universe: we begin with the empty game, denoted 0 or $\{\,|\,\}$, after which we may construct the games $\{0\,|\,0\}$, $\{0\,|\,\}$ and $\{\,|\,0\}$ (denoted $*, 1, -1$ respectively); then we make $2 = \{0, 1\,|\,\}$, $-2 = \{\,|\,0, -1\}$, etc., and so on. At limit ordinals we take the collection of all previously constructed games; ranging over all ordinals we obtain a universe of games much like a universe of (wellfounded) sets.

The two sides of a game are called left and right; thus a game x can contain a game y on the left, in which we write $y \in_{\mathrm{L}} x$, or x may contain y on the right, denoted $y \in_{\mathrm{R}} x$. Of course it is also possible that both are true. A typical left- or right-element of x is often denoted x^{L} or x^{R} respectively.

These two-sided containers are regarded as games (formally we define a game *on* them) as follows. Two players, often called Left and Right, alternate in choosing an element from their side of the current container. The next player must then choose an element, from his respective side, of his opponent's chosen container. The first player unable to move loses, and play always terminates in finite time since each game is wellfounded. Furthermore, each game is determined.

Conway games have a natural preorder, defined by

$$x \leq y \Leftrightarrow \text{ no } x^{\mathrm{L}} \geq y \text{ and no } y^{\mathrm{R}} \leq x$$

(that is, when we observe the normal play convention; with misère play the situation becomes more complicated, losing both transitivity and reflexivity of this order. Since misère play is much more complicated than normal play, we implicitly restrict attention to the latter). These games also have an interesting algebraic structure: for instance

addition is defined recursively by

$$x + y = \{x^{\mathrm{L}} + y,\, x + y^{\mathrm{L}} \,|\, x^{\mathrm{R}} + y,\, x + y^{\mathrm{R}}\},$$

and a negation by $-x = \{-x^{\mathrm{R}} \mid -x^{\mathrm{L}}\}$. We remark that although $x + -x$ is rarely equal to the empty game 0, we do have $0 \leq x + -x \leq 0$. Furthermore, when we factor by the equivalence $\simeq$, defined by $x \simeq y \leftrightarrow x \leq y \leq x$, we obtain a (class-sized) group under this addition and negation.

The preorder $\leq$ reflects the existence of winning strategies for each player. Specifically,

- $x \leq y \Leftrightarrow$ Left wins $y - x$ playing second, or equivalently Right wins $x - y$ playing second.

- $x \not\leq y \Leftrightarrow$ Left wins $y - x$ playing first, or equivalently Right wins $x - y$ playing first.

An extensive theory, including the above, is illuminated in ONAG, Winning Ways, etc. However, some areas are left wanting further explanation or study. Principal among these, we feel, is the discussion of strategies (beyond their use in proving results regarding other structure). The reader is left to ponder what, precisely, constitutes a strategy (surely, the most significant aspect of playing a game), and there is no consideration of structures where strategies have anything more than an implicit role.

Also of interest is the notion of a game's "value" so often used (see ONAG [7, p.76]). Although its use is (rightly) justified by its simplification of both presentation and calculation, its significance from a general mathematical perspective is not explored. Such an explanation will aid understanding of what gives Conway games their distinguished structure.

Infinite games are given some treatment in ONAG and Winning Ways: for instance, loopy games (usually finite, but allowing infinite play), as well as the better behaved infinite wellfounded games. However little indication is given of whether these constructions can be used – and if so, how – to understand more complicated infinite games. This in particular is a worthwhile pursuit.

1.2 Some further developments

We give here a brief indication of more recent literature. Since our interest is mainly in the fundamental characteristics of collections of games, we do not recount the large quantity of available information regarding particular games (such as Domineering, Hackenbush or Toads and Frogs). Neither will we be looking in any specific detail at the surreal numbers.

More recent literature does, rather satisfactorily, address strategies for games in their own right, using categories. The consideration of strategies in this context was first addressed by André Joyal in his 1977 paper *Remarques sur la théorie des jeux à deux personnes* [18][1]. He defines an arrow $f\colon G \to H$ as a winning strategy for the Left player, moving second, in the game $H - G$. If $f\colon G \to H$ and $g\colon H \to K$ then the composition

[1] We are grateful to Robin Houston for providing an English translation in 2003 [19]. Henceforth all references, in particular page numbers, are with respect to this version.

$g \circ f \colon G \to K$ is the strategy obtained by playing in $K - G$, via f, g and the swivelchair strategy in $H - H$[1]. Under these specifications the class of Conway games forms a compact closed category.

Building on this, Cockett et al. have introduced the concept of *combinatorial game category* [6] (as in their paper we occasionally shorten this to *cgc*). Several significant additions are made there to form the definition of a combinatorial game category, which it will be useful to describe here. Perhaps most importantly there is a notion of strategy for the Left player playing *first*, captured by a *module*. That is, the first-player strategies compose with second-player strategies on either side, though not necessarily with other first-player strategies.

Their definition also requires that a combinatorial game category is closed under a *diproduct* operation, which allows one to form the game

$$\{x_0, \ldots, x_n \,|\, y_0, \ldots, y_n\}$$

given games $x_0, \ldots, x_n, y_0, \ldots y_n$. Certain strategies are guaranteed for diproducts, by the *injection*, *projection* and *dituple* rules: for example, if y is a left member of z then the existence of a strategy $f \colon x \to y$ implies that of a first-player strategy $g \colon x \nrightarrow z$ (we follow their practice of distinguishing module arrows/first-player strategies by using a dashed arrow, $\nrightarrow$). Cockett et al. then demonstrate that the (hereditarily finite, or *short*) Conway games form an initial object in the category **cgc** of combinatorial game categories.

Strategies are also discussed more openly in *Conway Games, Coalgebraically*; Honsell and Lenisa discuss *hypergames*, which they define as the final coalgebra of a functor F on the category of classes, defined by

$$F(C) = \mathcal{P}(C) \times \mathcal{P}(C).$$

This contrasts with the Conway games, which form the initial algebra for F. This definition has the advantage of being simple, while also delivering certain structure (such as coinduction principles); however for our purposes, in some sense too many objects are included. In particular, the inclusion of games such as $x = \{x \,|\, x\}$ (coupled with an adherence to the rules governing formation of strategies used in ONAG and Winning Ways) forces one to look at non-losing strategies rather than winning strategies. Thus one must abandon the transitivity of second-player favourability, i.e. $\leq$. Still, this is a worthwhile generalisation which gives some indication of the difficulties involved with non-wellfounded and loopy games.

[1] The sum strategy $f + g$ is formed first. According to $f + g$ Left follows in whichever component Right's previous move was made, according ot the corresponding strategy (using f in $H - G$ and g in $K - H$). Since Left has a copycat strategy in $H - H$, we can eliminate all reference to H and $-H$, obtaining a strategy $g \circ f \colon G \to K$. For a more thorough explanation see Joyal [19, pp.3,4].

1.3 Material covered here

What we cover here is intended to address some of these issues, while also producing generalisations which will aid further research. In chapter 2 we look at the abstraction of Conway games in the context of preordered groups. Specifically we define a notion of two-ordered structure, which represents an arbitrary class of "games". Such collections are equipped with two orders representing favourability for a player moving first and second respectively, the latter forming a preorder (such might be expected, since we favour the normal play convention). We also require that the orders compose, in the same way that Conway's $\leq$ and $\lhd\!\!\mid$ compose.

Two-ordered groups, or togs for short, are then introduced as groups for which the underlying objects form a two-ordered structure compatible with the group multiplication. We discuss an elementary theory of these objects, including quotients, morphisms and products. The class of values of Conway games is a particular example of such a structure.

The following chapter introduces the notion of *game category*, a concept similar to the cgcs of Cockett et al. We remark that the presentation of first-player strategies given here is essentially the same as presented in their paper [6]; the instructive and constructive axioms derive from the discussion of strategies in Joyal [19], and are therefore analogous to the injection, projection and ditupling rules of their combinatorial game logic). They are used here in a different way, however: most significantly we are concerned primarily with set-theoretic aspects of combinatorial games, rather than category-theoretic properties. We do not require closure under a diproduct operation. Building on this we discuss additional structure such as tensor products.

Such generalisation does have its uses. In particular we are able to introduce a general value map, which acts as a functor from a collection of monoidal game categories to the space of two-ordered groups; Conway games and their values are one example of this construction.

The material presented in the remaining chapters is part of ongoing research; we hope to use nonstandard Conway games to illuminate areas in the theory of non-wellfounded, infinite games. To this aim we develop a set-like foundational theory of combinatorial games in chapter 4. This in itself is straightforward, and we proceed to show that the theory, called amphi-ZF, is synonymous with regular ZF (in the sense of Visser [34]): each theory interprets the other via a given interpretation, and moreover the two interpretations are inverse to one another. We remark that, although Conway and others have expressed certainty regarding the theories' equiconsistency[1], this result is stronger still; such a relationship is intentional, and demonstrates that amphi-ZF is really a rebranding of ZF within a language having two memberships, where objects with a single containment become objects with two separate notions of containment.

Finally in chapter 5 we look at a particular construction, within some fragment of ZF, with the aim of building an illfounded class of sets having useful topological properties (ZF, rather than ZF_2, was chosen since the problem became simpler for a one-sided set theory; the intention is to use similar techniques in the two-sided set theory). Although the construction differs from the topological set theory of (in particular) Forti et al. [14,

[1]See, for example, ONAG [7, p.66]. Surprisingly, Conway is uninterested in such exercises [7, pp. 66-7], despite expressing a desire to see more genetic definitions (essentially, recursive, set-theoretic definitions) in the study of Surreal numbers [7, pp.225-6].

12, 13, 15], the results are similar; therefore some comparison of the two approaches has been included. There is still scope, it seems, to draw parallels between the two; in particular the author is interested to know whether hypotheses of Forti et al. involving large cardinal axioms can be replaced by, for instance, a certain level of saturation in our nonstandard model. For more discussion, see chapter 5.

Once we have completed the construction, a discussion of the topological and set theoretic structures is given. In particular the new class of sets is proven to be extensional, and to allow as much "construction" as possible, although separation appears to be more problematic (a consequence of our nonstandard construction; this does not hinder us much in the context of game theory, however). The space is also shown to carry a metric compatible with the membership, in the sense that it is equal to the Hausdorff pseudometric obtained on the same space (this too can be compared with the constructions of Forti et al. [14, for example]). Further, assuming sufficient saturation we can show the space to be complete. Finally we discuss this construction in terms of the games of chapter 3.

CHAPTER 2

TWO-ORDERED STRUCTURES

Here we isolate and abstract the key order- and group-theoretic properties of Conway games; with this in mind discuss the elementary theory of *two-ordered groups*, defined below. The axioms we choose reflect the fundamental properties of Conway's orders $\leq$ and $\lhd\!\mid$.

2.1 Two-ordered groups

Definition 2.1.1. A *two-ordered group* is a group G with preorder $\leq$ and binary relation $\lhd\!\mid$, satisfying

T1 $\forall x, y, z \in G\ (x \leq y \to zx \leq zy \wedge xz \leq yz)$;

T2 $\forall x, y, z \in G\ (x \lhd\!\mid y \to zx \lhd\!\mid zy \wedge xz \lhd\!\mid yz)$;

T3 $\forall x, y, z \in G\ \big((x \lhd\!\mid y \wedge y \leq z) \vee (x \leq y \wedge y \lhd\!\mid z) \to x \lhd\!\mid z\big)$;

T4 $\forall x, y \in G\ (x \leq y \to y \not\lhd\!\mid x)$.

For brevity's sake we occasionally shorten "two-ordered group" to "tog".

Axioms T1 and T2 ensure that each order is compatible with the group multiplication. The axiom T3 implies that the two orders $\leq$, $\lhd\!\mid$ are composable in the usual way (see ONAG [7], for example). The final point is perhaps in need of justification. Axiom T4 reflects the fact that in many situations under the normal play convention[1], the second player cannot ever lose a game of the form $x^{-1}x$. Therefore the first player cannot win $x^{-1}x$; from this and the other assumptions we deduce that $x \not\lhd\!\mid x$, and hence axiom T4.

There are also good reasons – dependent on the situation – to drop axiom T4. Most significant among these is the fact that many illfounded games contradict this axiom (assuming a natural connection between the existence of strategies and availability of moves; see definition 3.4.1), and therefore rigidly assuming T4 may be a hindrance. Our

[1]Under so-called "normal play" the first player without a move loses (see ONAG [7, p.71]). Under this convention a player cannot lose the sum of a game with its opposite, since he always has the option of returning his opponent's move in the opposite game. Effectively the opponent is then forced to play against himself, and will necessarily run out of moves first (if the game ends at all). This is called the *copycat* strategy.

arguments in favour of T4 also depend on the strategies involved being winning strategies. This could be an unnecessary restriction in future. For now we keep the axiom (its inclusion does not affect any results here), although this may change in future.

Remarks 2.1.2.

- Often we will write $x \simeq y$ for $x \leq y \wedge y \leq x$. Since $\leq$ is a preorder compatible with G, $\simeq$ is an equivalence.
- Any group can be made into a tog; the simplest example is to take a trivial preorder $\leq$ on G (either $\forall x, y\,(x \leq y)$, or $\forall x, y\,(x \leq y \leftrightarrow x = y)$), and for $\mathrel{\lhd\!\shortmid}$ to be the empty relation.
- If G is already equipped with an appropriate preorder then we may take either $\mathrel{\lhd\!\shortmid} = \varnothing$, or more interestingly we can define $\mathrel{\lhd\!\shortmid}$ to be the relation $\not\geq$. It is easily checked that under these definitions G is a tog. This provides us with a great variety of nontrivial examples; these will help in understanding the following material.
- Notice that any subgroup H of the two-ordered group G will automatically be a tog under the inherited relations.

Positive cones

It is useful to have specific notation for the classes $\{x\colon x \geq 1\}$ and $\{x\colon x \mathrel{\shortmid\!\rhd} 1\}$. We define, for any two-ordered group G, the *positive cones*

$$\begin{aligned} P = P_G &= \{x \in G\colon x \geq 1\}; \\ Q = Q_G &= \{x \in G\colon x \mathrel{\shortmid\!\rhd} 1\}. \end{aligned}$$

It is well known that for a group G, the compatible preorders on G correspond to the collection of normal submonoids. We can show the following.

Proposition 2.1.3. Let $P, Q \subseteq G$ and define binary relation $\leq$, $\mathrel{\lhd\!\shortmid}$ on G by

$$\begin{aligned} x \leq y &\leftrightarrow yx^{-1} \in P; \\ x \mathrel{\lhd\!\shortmid} y &\leftrightarrow yx^{-1} \in Q. \end{aligned}$$

Then $(G, \leq, \mathrel{\lhd\!\shortmid})$ is a two-ordered group if and only if

- P is a normal submonoid of G;
- Q is a normal subset of G not containing 1;
- $PQ = QP = Q$.

Proof. The normality of P and Q is equivalent to the compatibility of the corresponding order with the multiplication in G. The transitivity of $\leq$ and the closure of P under multiplication are equivalent, and $1 \leq 1$ if and only if $1 \in P$. If G is a two-ordered group then the first two conditions are true, and so the axiom T3 implies $PQ = QP = Q$ as $1 \in P$. Conversely if the three conditions are satisfied then $x \leq y \mathrel{\lhd\!\shortmid} z$ if and only if $zy^{-1} \in Q$ and $yx^{-1} \in P$, hence $zx^{-1} \in Q$, and $x \mathrel{\lhd\!\shortmid} z$. Analogously, if $x \mathrel{\lhd\!\shortmid} y \leq z$ then $x \mathrel{\lhd\!\shortmid} z$, and therefore G is a tog. □

Definable closures

The following propositions will save us time in future. Let $\mathscr{L}_{\text{tos}}$ be the language of *two-ordered structures*, with binary relation symbols $\leq$, $\lhd\!\shortmid$ and constant symbol 1. We remark that T3 and T4 are sentences of $\mathscr{L}_{\text{tos}}$.

Proposition 2.1.4. Suppose G is a two-ordered group and for some $\mathscr{L}_{\text{tos}}$-formula $\phi(x)$ with free variable x, $S = \{x \in G \colon \phi(x)\}$. Then $S^G = S$ and $\langle S \rangle \trianglelefteq G$.

Proof. Suppose $\psi(x, \overline{y})$ is a quantifier-free $\mathscr{L}_{\text{tos}}$-formula. We prove by induction on the number of logical connectives in ψ that for all $x, \overline{y}, g$ in G, $\psi(x, \overline{y}) \leftrightarrow \psi(x^g, \overline{y}^g)$. An atomic such formula will be of the form uRv, where R is a binary relation among $\leq$, $\lhd\!\shortmid$, $=$; clearly the claim holds for these cases. If the claim is true of the formulas ψ_0 and ψ_1, then clearly $\psi_0(x, \overline{y}) \wedge \psi_1(x, \overline{y}), \psi_0(x, \overline{y}) \vee \psi_1(x, \overline{y}), \neg\psi_0(x, \overline{y})$ also satisfy. This proves the claim.

Now suppose that $\phi(x)$ defines the class S, and find a logically equivalent formula $\theta(x)$ which is in prenex normal form. Rename the bound variables so $\theta(x)$ becomes

$$\mathsf{Q}_0 y_0\ \mathsf{Q}_1 y_0 \ldots \mathsf{Q}_n y_n\ \psi(x, \overline{y}),$$

where the Q_i are quantifiers and ψ is quantifier free. If $g \in G$ is fixed then $\psi(x, y_0, \ldots, y_n)$ is true if and only if $\psi(x^g, y_0^g, \ldots, y_n^g)$. By considering the universal and existential cases separately, we see that, since conjugation by g is a bijection $G \to G$, $\mathsf{Q}_n y_n\ \psi(x, \overline{y})$ is equivalent to $\mathsf{Q}_n y_n\ \psi(x^g, y_0^g, \ldots, y_{n-1}^g, y_n)$. Proceeding in this way we can prove, by induction on the number of quantifiers in θ, that $\theta(x)$ is true if and only if $\theta(x^g)$. Since g was arbitrary and θ logically equivalent to ϕ, S is closed under conjugation. Since conjugation by g is a homomorphism $G \to G$ for each $g \in G$, it follows that $\langle S \rangle \trianglelefteq G$. $\square$

The same proof can apply to a similar proposition. Suppose we fix a tog G, and take as our language $\mathscr{L}_{\text{rm}}$ the relations $\leq$, $\lhd\!\shortmid$, along with unary function symbols r_g, representing right multiplication by g, for each $g \in G$, and constant symbol 1. Then any subset S of G which is definable by a unary $\mathscr{L}_{\text{rm}}$-formula is normal in G (i.e. closed under conjugation).

Aside from the cases $\mathscr{L}_{\text{tos}}, \mathscr{L}_{\text{rm}}$ there are still useful extensions to $\mathscr{L}_{\text{tos}}$ for which proposition 2.1.4 still holds. The obvious choices, however, fail. For instance we might consider the addition of an inversion or (binary) multiplication function symbol, or the scalar multiplication functions above with an identity. The following example demonstrates that in these cases proposition 2.1.4 fails.

Example 2.1.5. Let $G = D_6 = \langle x, y \colon x^3 = y^2;\ yxy = x^2 \rangle$, and consider $H = \langle y \rangle$. Since $xyx^{-1} = yx \notin H$, H is not normal in G. However, we can define H using the equalities

$$\begin{aligned} H &= \{g \colon g = g^{-1}\} \\ &= \{g \colon r_y(g) = 1 \vee g = 1\}, \end{aligned}$$

in the appropriate language $\mathscr{L}$. Furthermore, the set $\{g \colon \forall h(ggh = h)\} = \{1, y, xy, x^2y\}$ is not normal in G.

Although sets definable using multiplication and inversion are not necessarily normal, we can generalise proposition 2.1.4 as follows.

Proposition 2.1.6. Let $\mathscr{L}_S$ be the language $\mathscr{L}_{\text{tos}}$ with additional unary relation symbol S. Assume $X \subseteq G$ is normal, and that $X = \{g \in G \colon S(g)\}$. Then for any $\mathscr{L}_S$-formula $\phi(x)$, $\{g \in G \colon \phi(g)\}$ is normal in G.

Proof. We proceed as above. If $\psi(x, \overline{y})$ we prove that $G \vDash \psi(x, \overline{y}) \leftrightarrow \psi(x, \overline{y}^g)$ for all $g \in G$. Suppose $\psi(x, \overline{y})$ is an atomic $\mathscr{L}_S$-formula. In proposition 2.1.4 we covered all possible cases where ψ is an $\mathscr{L}_{\text{tos}}$-formula; we are left with the cases where ψ is $S(x)$ or $S(y)$. Since X is normal, the claim also holds in this case. The remainder of the proof is identical. □

Using this result we can consider a general notion of closure. Suppose $\phi(x)$ is an $\mathscr{L}_S$-formula with single free variable, and define $\phi_S(x)$ to be the formula obtained by replacing all occurrences of $S(v)$ in $\phi(x)$ by $\phi(v)$. So, for example, if $\phi(x)$ is $\exists y\,(S(y) \wedge y \leq x \wedge x \leq y)$, then ϕ^S is the formula $\exists y\,(\phi(y) \wedge y \leq x \wedge x \leq y)$. We will call ϕ a *closure formula* on G if, whenever $\{x \colon S(x)\}$ is normal,

$$\begin{aligned} G &\vDash \ \forall g\ (S(g) \to \phi(g)), \text{ and} \\ G &\vDash \ \forall g\ ((\phi(g) \leftrightarrow \phi^S(g))). \end{aligned}$$

If $C \colon \mathcal{P}(G) \to \mathcal{P}(G)$ is any function then we say C is a *definable closure* if and only if there is a closure formula ϕ such that whenever $X = \{x \in G \colon S(x)\}$ is normal, $C(X) = \{g \in G \colon \phi(g)\}$.

Notice that, by definition, definable closures are closures, in that $C(C(X)) = C(X)$ and $C(X) \supseteq X$ for all X.

In particular we define the following. The *null closure* of a subset X of G is

$$\overline{X} = \{y \in G \colon \exists x \in X\ y \simeq x\},$$

and the convex closure of X is

$$\widetilde{X} = \{y \in G \colon \exists x, x' \in X\ (x \leq y \leq x')\},$$

as usual. Notice that each is a definable closure, and so in particular each preserves normality of subsets of G. We will call a set X *exact* if $\overline{X} = X$, and (as usual) convex if $\widetilde{X} = X$. If $g \in G$ we define $C(g)$, for a closure C, to be $C(\{g\})$. The following results should be clear.

Proposition 2.1.7. Suppose G is a tog, $X \subseteq G$ and $K \subseteq G$ is a normal subgroup. Then

- G is exact if and only if $\overline{1} = \{1\}$, or equivalently if and only if $\leq$ is a partial order;
- $\overline{X} \subseteq \widetilde{X}$;
- $\overline{K}, \widetilde{K}$ are normal subgroups of G.

Proof. The first two points should be obvious; the last follows as null and convex closures are definable. □

2.2 Quotients

If we are interested in quotients of two-ordered groups we need to consider the related concepts of equivalence, morphism and normal substructure. There are various ways in which a quotient might be defined; we opt for a simple definition which is highlights the dual nature of $\leq$ and $\mathrel{\lhd\!|}$, but also makes every group quotient a quotient of togs. Let K be a normal subgroup of the tog G (clearly this is a necessary requirement), and attach the inherited orders $\leq, \mathrel{\lhd\!|}$. We know that these groups will factor nicely, so it remains to check that the quotient G/K inherits compatible orderings from G which preserve some tog structure. For all $x, y \in G$, set

$$xK \leq yK \leftrightarrow \exists k \in K\ xk \leq y; \tag{2.1}$$

$$xK \mathrel{\lhd\!|} yK \leftrightarrow \forall k \in K\ xk \mathrel{\lhd\!|} y. \tag{2.2}$$

Since K is normal in G the analogous conditions with xk replaced by kx are equivalent. If $xK \leq yK$ then some $k \in K$ satisfies $xk \leq y$, and so $xk \mathrel{|\!\not\rhd} y$, implying $xK \mathrel{\not\lhd\!|} yK$; that is, G/K satisfies T4. Since K is a group $\leq$ is a preorder on G/K. It should also be clear that the compatibility axioms (T1 and T2) are satisfied. If $xK \leq yK \mathrel{\lhd\!|} zK$ then for some fixed k_1 and all k_2 in K, $xk_1 \leq y$ and $yk_2 \mathrel{\lhd\!|} z$. Therefore $xk_1k_2 \leq yk_2 \mathrel{\lhd\!|} z$, implying $xk_1k_2 \mathrel{\lhd\!|} z$, for all $k_2 \in K$. If we are given $k \in K$, fix $k_2 = k - k_1 \in K$ so $xk = xk_1k_2 \mathrel{\lhd\!|} z$, and hence $xk \mathrel{\lhd\!|} z$. Since k was arbitrary, $xK \mathrel{\lhd\!|} zK$. The case $xK \mathrel{\lhd\!|} yK \leq zK$ is essentially the same.

Since a quotient can be defined whenever $K \trianglelefteqslant G$ (as groups), it makes sense to call any normal subgroup K endowed with the inherited orders *normal*, and write this simply as $K \trianglelefteqslant G$, using the standard notation from group theory.

Example 2.2.1. Let $G = \mathbb{Z}$ with the usual addition and ordering $\leq$, but with an empty relation $\mathrel{\lhd\!|}$. The only subgroups of $\mathbb{Z}$ are of the form $k\mathbb{Z}$. With the construction above, $\mathbb{Z}/k\mathbb{Z}$ inherits the trivial ordering: $x \simeq y$ for all x, y. Notice that if we were to reverse the quantifiers in our definitions of the inherited orders, we would have that no x satisfies $x \leq x$, and so $\leq$ would not be a preorder, and nor would $\mathbb{Z}/k\mathbb{Z}$ be a tog.

Proposition 2.2.2. Let G be a two-ordered group, and $K \trianglelefteqslant G$. Then

- $G/\overline{1}$ is exact;
- G/K is exact if and only if K is convex.

Proof. If $x\overline{1} \simeq \overline{1}$ there are $k_1, k_2 \simeq 1$ such that $1 \simeq k_1 \leq x \leq k_2 \simeq 1$; so $x \simeq 1$. For the second statement, notice that G/K is exact if and only if for all x,

$$xK \simeq K \Leftarrow x \in K. \tag{$\star$}$$

Since $xK \simeq K$ if and only if there exist $k_1, k_2 \in K$ such that $k_1 \leq x \leq k_2$, the implication $(\star)$ is equivalent to the statement that K is convex. $\square$

2.3 Morphisms

We consider two types of morphism here. The first, which we call *promorphisms*, are maps monotonic in both orders, and can be seen as functions preserving Left's advantage (if we interpret the order structure in the usual way). This notion of morphism is more straightforward, but the relationship with quotients is unsatisfactory. We later define *amphimorphisms*, which instead preserve advantage for the second player, and add no unnecessary advantage for the first. This definition allows more interesting structure while still being meaningful from the game-theoretic perspective.

2.3.1 Promorphisms

Definition 2.3.1. Suppose G, H are two-ordered groups. A group homomorphism $\varphi\colon G \to H$ is a *promorphism* if, for all $x, y \in G$,

$$x \leq y \longrightarrow \varphi x \leq \varphi y; \tag{2.3}$$

$$x \mathrel{\lhd\!\mid} y \longrightarrow \varphi x \mathrel{\lhd\!\mid} \varphi y. \tag{2.4}$$

A promorphism $\varphi\colon G \to H$ is a *preembedding* if $\varphi x \leq \varphi y$ implies $x \leq y$ and $\varphi x \mathrel{\lhd\!\mid} \varphi y$ implies $x \mathrel{\lhd\!\mid} y$. A preembedding which is also bijective is a *tog isomorphism*; an injective preembedding is simply an *embedding*.

A morphism theorem states that, given a morphism $\varphi\colon G \to H$, $G/\ker\varphi$ and $\operatorname{im}\varphi$ are isomorphic via a canonical isomorphism ψ, defined in terms of φ. We can prove a similar, weaker result for promorphisms.

Theorem 2.3.2 (Promorphism theorem)**.** Suppose G, H are two-ordered groups, and $\varphi\colon G \to H$ is a promorphism. Define $\psi\colon G/\ker\varphi \to \operatorname{im}\varphi$ by $\psi(x\ker\varphi) = \varphi x$ as usual. Then ψ is a (bijective) promorphism.

Moreover, φ is a preembedding if and only if both ψ is an isomorphism and $\ker\varphi = \overline{1}$.

Proof. We know that ψ is well-defined, and a group isomorphism; it remains to show ψ is a promorphism. Let $K = \ker\varphi$ and suppose $xK \leq yK$. Then for some $k \in K$, $xk \leq y$, so $\varphi x = \varphi(xk) \leq \varphi y$; that is, $\psi(xK) \leq \psi(yK)$. If $xK \mathrel{\lhd\!\mid} yK$ then in particular $x \mathrel{\lhd\!\mid} y$, so $\varphi x \mathrel{\lhd\!\mid} \varphi y$, and $\psi(xK) \mathrel{\lhd\!\mid} \psi(yK)$.

Suppose in addition φ is a preembedding. Then $\varphi x = 1$ implies $x \simeq 1$, so $\ker\varphi = \overline{1}$. If $\psi(xK) \leq \psi(yK)$, i.e. $\varphi x \leq \varphi y$, then $x \leq y$, so $xK \leq yK$. If $\psi(xK) \mathrel{\lhd\!\mid} \psi(yK)$ then $\varphi x \mathrel{\lhd\!\mid} \varphi y$, so $x \mathrel{\lhd\!\mid} y$. If $k \in K$ then $k \simeq 1$, so $xk \mathrel{\lhd\!\mid} y$. Therefore $xK \mathrel{\lhd\!\mid} yK$. This shows ψ is an isomorphism.

Conversely, assume ψ is an isomorphism and $\ker\varphi = \overline{1}$. If $\varphi x \leq \varphi y$ then $\psi(xK) \leq \psi(yK)$, so $xK \leq yK$. Since $K = \overline{1}$, this is equivalent to $x \leq y$. The same argument shows that $\varphi x \mathrel{\lhd\!\mid} \varphi y$ implies $x \mathrel{\lhd\!\mid} y$. Therefore φ is a preembedding. □

It may be that some reasonable condition equivalent to ψ being a preembedding can be found, weakening the hypothesis of theorem 2.3.2. However this definition of morphism also creates problems for projections onto quotients. We make the following definitions to clarify the situation.

Definition 2.3.3. A game k is called

- *neutral* or I-neutral if $\forall x \in G\ (x \mathrel{|\!\rhd} 1 \leftrightarrow x \mathrel{|\!\rhd} k)$;
- *null* (or perhaps II-neutral) if $\forall x \in G\ (x \geq 1 \leftrightarrow x \geq k)$;

If all the elements in a subclass H are null/neutral in G then we say H is null/neutral in G respectively.

A game k is T-neutral in G if and only if we cannot affect the T-strategies in any game when multiplying by k. We claim that the null and neutral games form normal subgroups of G.

Proposition 2.3.4. Let

$$\begin{aligned} \mathrm{Null}(G) &= \{x \colon x \text{ is null}\}; \\ \mathrm{Neu}(G) &= \{x \colon x \text{ is neutral}\}. \end{aligned}$$

Then $\mathrm{Null}(G)$ and $\mathrm{Neu}(G)$ are normal subgroups, with $\mathrm{Null}(G) = \overline{1}$.

Proof. Suppose k is null. Then $k \geq k$, so $k \geq 1$. If $x \geq k^{-1}$ implies $kx \geq 1$, and so $kx \geq k$ by nullity of k; hence $x \geq 1$. As $k \geq 1$, if $x \geq 1$ then $x \geq k^{-1}$. So k^{-1} is also null. In particular $k^{-1} \geq 1$ also, so $k \simeq 1$. Clearly every $k' \in \overline{1}$ is null; therefore $\mathrm{Null}(G) = \overline{1} \trianglelefteq G$.

If k is neutral, then $x \mathrel{|\!\rhd} k^{-1}$ is equivalent to $kx \mathrel{|\!\rhd} 1$, so to $kx \mathrel{|\!\rhd} k$, and hence to $x \mathrel{|\!\rhd} 1$. Therefore k is neutral if and only if k^{-1} is neutral. Since $\mathrm{Neu}\, G$ is clearly $\mathscr{L}_{\mathrm{tos}}$-definable, it is normal in G. Further, since this class is also closed under inverses and clearly contains 1, it is a normal subgroup of G. □

Proposition 2.3.5. Let K be a subgroup of the tog G. Then K is normal if and only if K is the kernel of a preordered group morphism $(G, \leq) \to (H, \leq)$. If we know K is normal in G, then

- K is neutral if and only if the canonical projection $\pi \colon G \to G/K$ is a promorphism;
- K is null if and only if the canonical projection $\pi \colon G \to G/K$ is a preembedding.

Proof. If $K \trianglelefteq G$ as groups then K is the kernel of the projection $\pi \colon G \to G/K$. By our definition of quotient, this projection is also a morphism of preordered groups. Clearly the converse is true: if $\phi \colon G \to H$ is a preordered group morphism, then $\ker \phi$ is normal.

For the next point, we show that π is $\lhd\!|$-monotonic if and only if K is neutral. If π is $\lhd\!|$-monotonic and $1 \mathrel{\lhd\!|} x$ then $K \mathrel{\lhd\!|} xK$, whence $k \mathrel{\lhd\!|} x$ for all $k \in K$. If instead $x \mathrel{|\!\rhd} k \in K$ then $xK \mathrel{|\!\rhd} K$, whence $x \mathrel{|\!\rhd} 1$. Conversely, if K is neutral, then for $k \in K$, $k \mathrel{\lhd\!|} x$ is equivalent to $1 \mathrel{\lhd\!|} x$, and so π is monotonic.

For the final point, if $K \trianglelefteq G$ is null then $x \mathrel{\lhd\!|} y$ if and only if $\pi x \mathrel{\lhd\!|} \pi y$, for all $x, y \in G$. Since nullity implies neutrality, π is a promorphism. If $xK \leq yK$ then for some $k \in K$, $xk \leq y$; but then $x \leq y$ since $k \simeq 1$. Hence π is a preembedding. If, conversely, π is a preembedding and $k \in K$, then $\pi(k) = 1$ so $k \simeq 1$. □

This result, along with the weak nature of our morphism theorem, suggests that an alternative may be preferable. We choose to look at morphisms which exploit the apparent duality between $\leq$ and $\lhd\!|$, but are also fairer to Right.

2.3.2 Amphimorphisms

Definition 2.3.6. Let G, H be two-ordered groups, and φ a homomorphism $G \to H$. We call φ an *amphimorphism* if both

$$\text{whenever } x \le y \text{ in } G,\ \varphi x \le \varphi y; \tag{2.5}$$

$$\text{whenever } \varphi x \mathrel{\lhd\!|} \varphi y \text{ in } H,\ x \mathrel{\lhd\!|} y. \tag{2.6}$$

Notice that if II prefers y to x, i.e. $y \ge x$ (if Left plays second) or $y \le x$ (if Right plays second) then this preference is preserved by every amphimorphism φ. Further, if I does not prefer y to x, i.e. $y \not\mathrel{|\!\rhd} x$ or $x \not\mathrel{|\!\rhd} y$, then I has no preference in the image of φ. So favourability for II is preserved, while that for I is at worst the same. That is, these morphisms favour the second player. Notice that amphimorphisms can be composed to form amphimorphisms.

For any function φ and any class S we let $\varphi[S]$ denote the class $\{\varphi(s) : s \in S\}$.

Proposition 2.3.7. If $\varphi\colon G \to H$ is a group homomorphism then φ is an amphimorphism if and only if

1. $\varphi[P_G] \subseteq P_H$;
2. $\varphi^{-1}[Q_H] \subseteq Q_G$.

As with promorphisms, given an amphimorphism $\varphi\colon G \to H$ we can prove that the induced group isomorphism $\psi\colon G/\ker\varphi \to \operatorname{im}\varphi$ is an amphimorphism. Later we will prove a more detailed result.

Theorem 2.3.8 (Amphimorphism theorem, part 1)**.** Suppose $\varphi\colon G \to H$ is an amphimorphism. Then the induced map $\psi\colon G/\ker\varphi \to \operatorname{im}\varphi$ is an amphimorphism. Furthermore φ is a preembedding if and only if ψ is an isomorphism and $\ker\varphi$ is null.

Proof. In light of the analogous theorem for promorphisms, we need only prove ψ preserves $\mathrel{\lhd\!|}$ in the backwards direction. Let $K = \ker\varphi$, and suppose $xK \not\mathrel{\lhd\!|} yK$. Then there exists $k \in K$ such that $xk \not\mathrel{\lhd\!|} y$. Since φ is an amphimorphism $\varphi x = \varphi(xk) \not\mathrel{\lhd\!|} \varphi y$, i.e. $\psi(xK) \not\mathrel{\lhd\!|} \psi(yK)$. □

We can also show that projections are amphimorphisms.

Proposition 2.3.9. Suppose $K \trianglelefteqslant G$ and $\pi\colon G \to G/K$ is the canonical projection. Then π is an amphimorphism.

Proof. We have seen that π necessarily preserves $\le$ in the forwards direction. If $xK \mathrel{\lhd\!|} yK$ then in particular $x \mathrel{\lhd\!|} y$, so π is an amphimorphism. □

We aim to extend theorem 2.3.8 by identifying precisely when ψ is an isomorphism. With this in mind we make the following definitions.

Definition 2.3.10. Let $\varphi\colon G \to H$ be an amphimorphism. We call φ

- $\le$*-good* if for all $h \in \operatorname{im}\varphi$, if $h \ge 1$ there is $g \in G$ such that $g \ge 1$ and $\varphi g = h$;
- $\mathrel{\lhd\!|}$ *-good* if for all $h \in \operatorname{im}\varphi$, if $h \not\mathrel{|\!\rhd} 1$ there is $g \in G$ such that $g \not\mathrel{|\!\rhd} 1$ and $\varphi g = h$;

- *good* if φ is $\leq$-good and $\lhd\!\shortmid$-good.

We can concisely state these definitions as

- φ is $\leq$-good if $P_{\operatorname{im}\varphi} \subseteq \varphi[P_G]$;
- φ is $\lhd\!\shortmid$-good if $Q^c_{\operatorname{im}\varphi} \subseteq \varphi[Q^c_G]$.

Proposition 2.3.11. Suppose $\varphi\colon G \to H$ is an amphimorphism, and $K = \ker\varphi$. Then φ is

1. $\leq$-good if and only if
$$\forall x, y \in G\ (\varphi x \leq \varphi y \to xK \leq yK); \tag{†}$$
2. $\lhd\!\shortmid$-good if and only if
$$\forall x, y \in G\ (xK \lhd\!\shortmid yK \to \varphi x \lhd\!\shortmid \varphi y). \tag{‡}$$

Proof. Assume φ is $\leq$-good, and $\varphi x \leq \varphi y$; then $1 \leq \varphi(yx^{-1})$, so there is $g \in P_G$ such that $\varphi g = \varphi(yx^{-1})$. Hence $g^{-1}yx^{-1} \in K$, and
$$xK = (g^{-1}yx^{-1})K\,xK = g^{-1}yK.$$
As $g^{-1} \leq 1$, $g^{-1}K \leq K$ and so $xK = g^{-1}yK \leq yK$. This proves (†).

Conversely, suppose (†) holds and $\varphi u \geq 1$. Then by (†) $uK \geq K$, i.e. there is a $k \in K$ such that $u \geq k$. Therefore $uk^{-1} \geq 1$, and $\varphi(uk^{-1}) = \varphi u$.

To see the second equivalence, proceed similarly. If φ is $\lhd\!\shortmid$-good and $\varphi x \not\lhd\!\shortmid \varphi y$ in H, $\varphi(yx^{-1}) \shortmid\!\not\rhd 1$, so there is $g \in G$ such that $g \shortmid\!\not\rhd 1$ and $\varphi g = \varphi(yx^{-1})$, proving (‡).

Conversely if (‡) is true and $\varphi u \shortmid\!\not\rhd 1$, then $uK \shortmid\!\not\rhd K$, so for some $k \in K$, $u \shortmid\!\not\rhd k$. Therefore $uk^{-1} \shortmid\!\not\rhd 1$, and $\varphi(uk^{-1}) = \varphi u$. □

Immediately we can extend our morphism theorem.

Theorem 2.3.12. Suppose $\varphi\colon G \to H$ is an amphimorphism. Then the induced group isomorphism $\psi\colon G/\ker\varphi$ is an isomorphism of togs if and only if φ is good.

There are various ways of closing the kernel $\ker\varphi$ for a morphism $\varphi\colon G \to H$. A relatively useful closure, which is not generally definable without adding a function symbol to our language, is given by
$$\ker_{\simeq}\varphi = \varphi^{-1}\left[\overline{1}\right].$$
Notice that
$$\ker\varphi \leq \overline{\ker\varphi} \leq \widetilde{\ker\varphi} \leq \ker_{\simeq}\varphi,$$
and that, for example, if φ is a preembedding then $\overline{\ker\varphi} = \ker_{\simeq}\varphi$.

Another direction we might take in extending our morphism theorem is to alter the induced morphism ψ. Suppose that G, H are groups and $\varphi\colon G \to H$, $K = \ker\varphi$. If C denotes any kind of closure which preserves normality, then $CK \trianglelefteq G$ and $CK/K \trianglelefteq G/K$.

Therefore CK/K has an isomorphic image $N \trianglelefteqslant \operatorname{im}\varphi$. By the isomorphism theorems, there is an isomorphism

$$\psi_C \colon G/CK \cong (G/K)/(CK/K) \cong \operatorname{im}\varphi/N.$$

It may be interesting to consider whether, under appropriate restrictions, such a construction might provide a tog isomorphism, for some C.

Question 2.3.13. Suppose L' is an appropriate 2-sorted language with constant symbol 1, unary function symbol f, unary relation symbol S, and binary relation symbols $\leq$, $\lhd\!\shortmid$. Let T be the theory stating that the elements of each sort form two-ordered groups, and that f is an amphimorphism between these togs. Is there a nontrivial, or minimal, formula ϕ in L' such that T proves both

- ϕ is a closure formula;
- the group isomorphism ψ_C described above is necessarily an isomorphism of togs?

2.4 Automorphisms of two-ordered structures

Groups of (pro- or amphi-) morphisms are also equipped with compatible two-orders. Clearly the composition of two morphisms is also a morphism. Let $\operatorname{Aut} G$ denote the class of tog automorphisms of G (that is, the bijective preembeddings $G \to G$). We now consider general two-ordered groups as collections of automorphisms.

Definition 2.4.1. Let S be any class, and assume that orders $\leq$ and $\lhd\!\shortmid$ are defined on S such that $\leq$ is a preorder and $(S, \leq, \lhd\!\shortmid)$ satisfies the axioms T3 and T4. Then $(S, \leq, \lhd\!\shortmid)$ is called a *two-ordered structure*.

We can define morphisms between two-ordered structures analogously to those for two-ordered groups, by dropping the group structure. That is, a promorphism is a map monotonic in both orders, while an amphimorphism is monotonic in one and reverse-monotonic in the other (as above). A preembedding is a map where each of these implications is an equivalence, and an automorphism is a bijective preembedding. As usual the automorphism space of a two-ordered structure will be denoted $\operatorname{Aut} S$.

If $\varphi, \psi \colon S \to T$ are morphisms of two-ordered structures, we write

$$\varphi \leq \psi \Leftrightarrow \forall s \in S\, (\varphi s \leq \psi s);$$
$$\varphi \lhd\!\shortmid \psi \Leftrightarrow \exists s \in S\, (\varphi s \lhd\!\shortmid \psi s).$$

Proposition 2.4.2. If S is a two-ordered structure, then $\operatorname{Aut} S$ is a two-ordered group with respect to these relations.

Proof. Clearly $\operatorname{Aut} S$ is a group; we show that the composition operation is compatible with the orders $\leq$, $\lhd\!\shortmid$, and that these define a two-ordered structure.

Suppose that $\varphi, \psi, \vartheta \in \operatorname{Aut} S$ with $\varphi \leq \psi$. If $s \in S$, we have $\varphi(\vartheta s) \leq \psi(\vartheta s)$, so $\varphi \circ \vartheta \leq \psi \circ \vartheta$. Also $\varphi s \leq \psi s$ so $\vartheta\varphi s \leq \vartheta\psi s$; hence $\operatorname{Aut} S$ satisfies axiom T1.

If $\varphi \lhd \psi$ there is $s \in S$ such that $\varphi s \lhd \psi s$; hence $\vartheta \circ \varphi(s) \lhd \vartheta \circ \psi(s)$. Also, if $t = \vartheta^{-1}s$, then $\varphi \circ \vartheta(t) = \varphi s \lhd \psi s = \psi \circ \vartheta(t)$. Therefore $\operatorname{Aut} S$ satisfies axiom T2.

If $\varphi \lhd \psi \leq \vartheta$, take $s \in S$ such that $\varphi s \lhd \psi s$; then since $\psi s \leq \vartheta s$, $\varphi s \lhd \vartheta s$ and $\varphi \lhd \vartheta$. Similarly we can prove that $\varphi \lhd \psi \leq \vartheta$ implies $\varphi \lhd \vartheta$, and so $\operatorname{Aut} S$ satisfies T3.

Finally, if $\varphi \leq \vartheta$ then given any $s \in S$, $\varphi s \leq \vartheta s$ and so $\varphi s \not\rhd \vartheta s$, implying T4. $\square$

Remark 2.4.3. Here we could replace the existential quantifier with a universal one, without affecting the above result. For simplicity here we keep a single definition; however, in future we may prefer to (for example) use the $\forall$-variant for promorphisms.

Theorem 2.4.4. Let G be a two-ordered group. Then the usual embedding, ϑ, which takes $g \in G$ to the map defined by $\vartheta g(x) = xg$, is a tog embedding. In particular every two-ordered group arises as an automorphism group for some two-ordered structure.

Proof. Indeed, $\vartheta g \leq \vartheta h$ if and only if $xg \leq xh$ for all $x \in G$, a condition equivalent to $g \leq h$. Similarly $\vartheta g \lhd \vartheta h$ is equivalent to the existence of some $x \in G$ such that $xg \lhd xh$, and so to $g \lhd h$. $\square$

2.5 Duality and determinacy

Much of the material covered thus far hints at some form of duality between the orders $\leq$ and $\lhd$. In this section we show that this is indeed the case, and explicitly describe how the duality works and can be used.

Many statements above regarding one order are accompanied by a similar statement regarding the other. The duality is not always clear, since at times (for example) quantifiers are reversed, and at others they remain the same. This is because the dual $\phi^\star$ of a first-order formula ϕ is obtained by swapping and negating relations as follows. Assume $\mathscr{L}$ is any first-order language containing the binary relation symbols $\leq$ and $\lhd$, and $\phi(\overline{v})$ an $\mathscr{L}$-formula. Then $\phi^\star(\overline{v})$ is obtained by swapping occurrences of $\leq$ or $\lhd$ with $\not\lhd$ or $\not\leq$ respectively. The following points make this explicit.

- If $\phi(\overline{v})$ is $t_0(\overline{v}) \leq t_1(\overline{v})$ or $t_0(\overline{v}) \lhd t_1(\overline{v})$, where the t_i represent terms, then $\phi^\star(\overline{v})$ is $t_0(\overline{v}) \not\lhd t_1(\overline{v})$ or $t_0(\overline{v}) \not\leq t_1(\overline{v})$ respectively. If $\phi(\overline{v})$ is $t_0(\overline{v}) = t_1(\overline{v})$ then $\phi^\star$ is ϕ.
- If $\phi(\overline{v})$ is $\psi_0(\overline{v}) \wedge \psi_1(\overline{v})$ or $\neg\psi_0(\overline{v})$, then $\phi^\star(\overline{v})$ is $\psi_0^\star(\overline{v}) \wedge \psi_1^\star(\overline{v})$ or $\neg\psi_0(\overline{v})$ respectively.
- If $\phi(\overline{v})$ is $\exists w\ \psi(\overline{v}, w)$, then $\phi^\star(\overline{v})$ is $\exists w\ \psi^\star(\overline{v}, w)$.

Examples 2.5.1.

- Suppose G is a two-ordered group and $K \trianglelefteq G$. We work in an appropriately expanded language $\mathscr{L}$. Recall $xK \leq yK$ if and only if $\exists k \in K\ (xk \leq y)$; the dual of this statement is
$$\exists k \in K\ (xk \not\lhd y); \tag{$\dagger$}$$
if we define $\lhd$ on G/K by declaring that $xK \not\lhd yK$ if and only if ($\dagger$) holds, then we get the original definition of $\lhd$, since ($\dagger$) is equivalent to the negation of the usual definition.

- Suppose G, H are togs with $\varphi\colon G \to H$ any function, and we work in an appropriate 2-sorted language, with a function symbol which we interpret as φ. For φ to be an amphimorphism, we require both

$$\forall x, y \in G\ (x \leq y \to \varphi x \leq \varphi y); \tag{2.7}$$

$$\forall x, y \in G\ (\varphi x \mathrel{\triangleleft\!|} \varphi y \to x \mathrel{\triangleleft\!|} y). \tag{2.8}$$

  The second statement is clearly equivalent to the sentence

$$\forall x, y \in G\ (x \mathrel{\ntriangleleft\!|} y \to \varphi x \mathrel{\ntriangleleft\!|} y),$$

  which is dual to statement (2.7) above.

- The formula $\phi(k)$ which states that k is neutral is clearly equivalent to the formula

$$\forall x \in G\ (1 \mathrel{\ntriangleleft\!|} x \leftrightarrow k \mathrel{\ntriangleleft\!|} x),$$

  say $\psi(k)$. Notice that $\psi^\star(k)$ is the statement that k is null.

- It should be clear that the notions of $\leq$-good and $\mathrel{\triangleleft\!|}$-good are also dual.

Notice that even the axioms of two-ordered groups exhibit some duality. To illustrate this point further we give an alternative axiomatisation to that given above, which is equivalent when G is a group:

T1* $\forall x, y, z \in G\ \big((x \leq y \leftrightarrow zx \leq zy) \wedge (x \leq y \leftrightarrow xz \leq yz)\big)$;

T2* $\forall x, y, z \in G\ \big((x \mathrel{\ntriangleleft\!|} y \leftrightarrow zx \mathrel{\ntriangleleft\!|} zy) \wedge (x \mathrel{\ntriangleleft\!|} y \leftrightarrow xz \mathrel{\ntriangleleft\!|} yz)\big)$;

T3 $\forall x, y, z \in G\ \big((x \mathrel{\triangleleft\!|} y \wedge y \leq z) \vee (x \leq y \wedge y \mathrel{\triangleleft\!|} z) \to x \mathrel{\triangleleft\!|} z\big)$;

T4* $\forall x \in G\ (x \mathrel{\ntriangleleft\!|} x)$;

T5 $\forall x, y, z \in G\ (x \leq y \wedge y \leq z \to x \leq z)$;

T6 $\forall x\ (x \leq x)$.

Notice that axioms T1*, T2*, T4* are equivalent to T1, T2, T4 respectively (since G is a group); the third is unchanged; and the remaining axioms state that $\leq$ is a preorder. We remark that T1* and T2* are clearly dual to one another, and also T4 is dual to T6, the statement that $\leq$ is reflexive.

We aim to show that the remaining axioms (T3 and T5) are also dual to statements worth consideration. First, however, we require the following definition.

Definition 2.5.2. Let S be a two-ordered group. An element g of G is *determined* if we have

$$(g \leq 1 \vee 1 \mathrel{\triangleleft\!|} g) \wedge (g \mathrel{\triangleleft\!|} 1 \vee 1 \leq g).$$

We denote the formula above by $\det(g)$.

We call a subclass U of G determined if every $u \in U$ is determined.

Given our usual interpretation of $\leq$ and $\mathrel{\lhd\!|}$, a determined game x is one for which one player will always be favoured, regardless of who moves first (although *which* player is favoured may depend on who moves first). Clearly such objects are of interest in game theory, and in particular we remark that the wellfounded games considered in ONAG and Winning Ways are determined. In fact, as we aim to show, this kind of determinacy is a natural concept to study in the logical and algebraic theory of two-ordered structures. In particular, we can prove the following. Let Det denote the statement $\forall x \ \det(x)$.

Proposition 2.5.3. Let G be a two-ordered group. Then $G \vDash \text{Det}$ if and only if $G \vDash \text{T3}^\star$

Proof. Suppose G is determined. If $x \not\mathrel{\lhd\!|} y \not\leq z$, i.e. $x \geq y \mathrel{|\!\rhd} z$, then $x \mathrel{|\!\rhd} z$ so $x \not\leq z$. Similarly $x \not\leq y \not\mathrel{\lhd\!|} z$ implies $x \not\leq z$, proving $\text{T3}^\star$.

Conversely if $G \vDash \text{T3}^\star$ and $x \not\leq 1 \not\mathrel{\lhd\!|} x$ in G, then $x \not\leq x$, a contradiction. Thus $x \leq 1$ or $1 \mathrel{\lhd\!|} x$. Similarly we prove $x \mathrel{\lhd\!|} 1 \vee 1 \leq x$, and so x is determined. □

Notice that the dual of T5 above is the statement that $\mathrel{\lhd\!|}$ is the complement of a transitive relation on G, which is implied by determinacy. An immediate consequence of this is that, if Th_{tog} denotes the theory of two-ordered groups, then $\text{Th}_{\text{tog}}+\text{Det}$ is self-dual, i.e. for all $\mathscr{L}_{\text{tog}}$-sentences σ, $\sigma \in \text{T}\textsc{og} \cup \{\forall x \ \det(x)\}$ if and only if $\sigma^\star \in \text{T}\textsc{og} \cup \{\forall x \ \det(x)\}$.

Question 2.5.4. Let $\mathscr{L}_{\text{tog}}$ be the language of two-ordered groups, i.e. $\mathscr{L}_{\text{tos}}$ with additional binary function symbol $\cdot$ and unary function symbol $^{-1}$. Assuming G is a determined two-ordered group, is $\text{Th}(G; \mathscr{L}_{\text{tog}})$, the theory of G in the language $\mathscr{L}_{\text{tog}}$, self-dual?

If the answer is positive, then the implication is an equivalence, by proposition 2.5.3.

CHAPTER 3

GAME CATEGORIES

Now we generalise the work of the previous chapter, using categories. As discussed in the introduction this allows us to consider strategies as distinguished entities. In particular we introduce a fundamental notion of *game category*, and attempt to show that the structures described within are important to the theory of these objects.

This work is related to the work of Cockett et al. [6] on Combinatorial Game Categories (cgcs); in fact our game categories form a strict subclass of the collection of module categories. There are many distinctions, however. For instance, while in their paper functors of module categories (that is, the arrows in **modcat**) correspond to what we have called promorphisms, we also look at maps more closely related to amphimorphisms, in an attempt to preserve a sense of duality as discussed in the previous chapter.

3.1 Game categories

It will be useful to fix some notation here. If A is a category then we denote its object class by $\mathrm{Obj}(A)$ and its arrow class by $\mathrm{Arr}(A)$; we consider A to be the pair $(\mathrm{Obj}(A), \mathrm{Arr}(A))$. We write $a \in A$ to mean $a \in \mathrm{Obj}(A)$, and $f : a \to b$ to mean $f \in \mathrm{Arr}(A)$ with domain $\operatorname{dom} f = a$ and codomain $\operatorname{cod} f = b$. Occasionally we will use Mac Lane's notation regarding monoidal categories: α for the associating transformation, and λ, ρ for the left- and right-identity transformations.

In many cases we will adjoin an additional arrow class $\mathrm{MArr}(A)$. These arrows will be denoted differently: if $g \in \mathrm{MArr}(A)$ has domain a and codomain b, we write $g : a \twoheadrightarrow b$. The class $\mathrm{MArr}(A)$ will always form a module[1] over the category $(\mathrm{Obj}(A), \mathrm{Arr}(A))$; therefore we refer to arrows from $\mathrm{MArr}(A)$ as *module arrows*. Since normal arrows (that is, arrows in $\mathrm{Arr}(A)$) are intended to represent second-player strategies between games (in the sense of Joyal [19], when there is sufficient structure), and module arrows represent first-player strategies (in the sense of Cockett et al. [6]), we will often refer to them as such.

[1] We use "module" in the sense of Cockett et al. [6]. Other terminology for these objects includes "profunctor" and "distributor"; see in particular the nlab page on profunctors [1]. Below we have chosen to define these objects explicitly for the sake of clarity. Further, in our definitions there is no reliance on the category of sets; since we frequently consider proper classes, we will also benefit by avoiding this limitation.

Definition 3.1.1. A *game category* is a category A, along with additional arrow class $\mathrm{MArr}(A)$ such that for all $a, b, c \in A$,

- there is no module arrow $f\colon a \nrightarrow a$;
- if $f\colon a \to b$ and $g\colon b \nrightarrow c$ there is a composite arrow $g \circ f\colon a \nrightarrow c$ in $\mathrm{MArr}(A)$;
- if $g\colon a \nrightarrow b$ and $f\colon b \to c$ there is a composite arrow $f \circ g\colon a \nrightarrow c$ in $\mathrm{MArr}(A)$;

and this composition satisfies the following associativity rules.

- If $f\colon a \to b$, $g\colon b \nrightarrow c$, and $h\colon c \to d$, then $(h \circ g) \circ f = h \circ (g \circ f)$ (central associativity);
- if $f'\colon a' \to a$, $f\colon a \to b$, and $g\colon b \nrightarrow c$ then $(g \circ f) \circ f' = g \circ (f \circ f')$ (left associativity);
- if $g\colon b \nrightarrow c$, $h\colon c \to d$, and $h'\colon d \to d'$ then $h' \circ (h \circ g) = (h' \circ h) \circ g$ (right associativity).

Since the composition of arrows in a game category is associative, we will avoid bracketing where the meaning of an expression is clear.

The definition we have given for game categories has various equivalent characterisations. In terms of module categories, a game category is precisely a module category A such that we never have $a \nrightarrow a$ for $a \in A$. Therefore a game category is an object in the category of module categories, **modcat**, of Cockett et al. If we ignore certain foundational issues then an equivalent definition is that A is enriched over some category S of sets and that there is a profunctor or distributor $\Phi\colon A^{\mathrm{op}} \times A \to S$, such that $\Phi(a, a)$ is always empty. In fact, this is essentially the definition of an enriched game category given below.

3.1.1 Promorphisms of game categories

As in chapter 2 we define two different types of morphism. In this case, the theory of amphimorphisms is still being developed. We begin with promorphisms.

Definition 3.1.2. Let A, B be game categories. A *promorphism* , or *pro-game functor*, from A to B is a functor $F\colon A \to B$ of the underlying categories, with the additional property that, for each module arrow $g\colon x \nrightarrow y$ in A, there is an assigned module arrow $Fg\colon Fx \nrightarrow Fy$ such that, whenever

$$w \xrightarrow{f} x \overset{g}{\nrightarrow} y \xrightarrow{h} z$$

in A, we have

$$F(h \circ g \circ f) = Fh \circ Fg \circ Ff$$

in B.

We let GC denote the collection of game categories, and consider promorphisms $F\colon A \to B$ as arrows between objects A, B in GC. It is easily checked that this forms a category, which we denote by $\mathrm{GC}^{\mathrm{PRO}}$.

Definition 3.1.3. Suppose $A, B \in \mathrm{GC}$ and $F, G\colon A \to B$ are promorphisms. A *natural transformation for promorphisms* or *pro-transformation* $\tau\colon F \to G$ is a natural transformation of the underlying functors such that, whenever $g\colon x \nrightarrow y$ in A, the diagram in figure 3.1 commutes.

$$\begin{tikzcd} Fx \arrow[r, "\tau_x"] \arrow[d, "Fg"', "|" marking] & Gx \arrow[d, "Gg", "|" marking] \\ Fy \arrow[r, "\tau_y"'] & Gy \end{tikzcd}$$

Figure 3.1: A natural transformation of promorphisms.

It is routine to show that the composition of such natural transformations is also a natural transformation of promorphisms; moreover this composition is associative, since the functions remain the same. Therefore for game categories A and B the homset $\mathrm{GC}^{\mathrm{PRO}}[A, B]$ is also a category.

There are many reasons for us to study a corresponding notion of module arrow in the category $\mathrm{GC}^{\mathrm{PRO}}[A, B]$; for instance, if we wish to embed a game category A onto some endomorphism space, this can only be done if there is a sensible notion of module transformation. This will become more apparent when we consider additional monoidal structure. As with two-ordered structures there are (at least) two sensible notions of module transformation. For the sake of simplicity here we choose the existential analogue.

Definition 3.1.4. Suppose $F, G\colon A \to B$ are promorphisms of game categories. A module transformation $\mu\colon F \nrightarrow G$ is a module arrow $\mu\colon Fx \nrightarrow Gx$, for some $x \in A$.

Proposition 3.1.5. Let $A, B \in \mathrm{GC}^{\mathrm{PRO}}$. Then $\mathrm{GC}^{\mathrm{PRO}}[A, B]$ is a game category, with natural transformations as arrows and module arrows as above.

Proof. To see $\mathrm{GC}^{\mathrm{PRO}}[A, B]$ is a category we must prove that transformations compose appropriately. Suppose

$$F \xrightarrow{\tau} G \xrightarrow{\eta} H,$$

and let $\eta \cdot \tau$ be the usual composite transformation. If $g\colon x \nrightarrow y$ in A then both squares of the rectangle in figure 3.2 commute, and hence the entire diagram commutes. Thus

$$\begin{tikzcd} Fx \arrow[r, "\tau_x"] \arrow[d, "Fg"', "|" marking] & Gx \arrow[r, "\eta_x"] \arrow[d, "Fg"', "|" marking] & Hx \arrow[d, "Hg", "|" marking] \\ Fy \arrow[r, "\tau_y"'] & Gy \arrow[r, "\tau_y"'] & Hy \end{tikzcd}$$

Figure 3.2: Composition of natural transformations in $\mathrm{GC}^{\mathrm{PRO}}$.

$\eta \cdot \tau$ is a natural transformation $F \to H$.

If we have

$$F \xrightarrow{\tau} G \overset{\mu}{\nrightarrow} H \xrightarrow{\eta} K$$

in $\mathrm{GC}^{\mathrm{PRO}}[A, B]$, and $f\colon x \to y$ in A, then the diagram in figure 3.3 commutes since each square commutes. Therefore if we define

$$(\eta \cdot \mu \cdot \tau)_x = \eta_x \circ \mu_x \circ \tau_x$$

for all $x \in A$, $\eta \cdot \mu \cdot \tau$ becomes a module arrow $F \twoheadrightarrow K$. Moreover it follows that all the associativity axioms hold, as an immediate consequence of the same axioms holding in A.

□

$$\begin{tikzcd}
Fx \arrow[r, "\tau_x"] \arrow[d, "Ff"'] & Gx \arrow[r, "\mu_x", "+"{marking}] \arrow[d, "Gf"'] & Hx \arrow[r, "\eta_x"] \arrow[d, "Hf"] & Kx \arrow[d, "Kf"] \\
Fy \arrow[r, "\tau_y"'] & Gy \arrow[r, "\mu_y"'] & Hy \arrow[r, "\eta_y"'] & Ky
\end{tikzcd}$$

Figure 3.3: Composition of normal and module arrows in $\mathrm{GC}^{\mathrm{PRO}}$.

Products in $\mathbf{GC^{pro}}$

Some discussion should be dedicated to products of game categories, since different constructions are required for different purposes. In some algebraic contexts, it is correct to assume that the obvious categorical product of two game categories A and B will suffice. That is, to use the usual category with pairs of arrows as morphisms, and taking pairs of module arrows as the product module arrows. However, as the next example demonstrates, this notion of product does not always give the most appropriate first-player structure.

Example 3.1.6. In a monoidal category M the tensor product $\otimes$ is taken to be a bifunctor from $M \times M$ to M, subject to certain constraints. This method does not achieve the desired result when applied to game categories. Consider, for example, a space $\mathscr{G}$ of partisan games containing 0 and 1 (cf. ONAG [7], Winning Ways [4]). If we are to view $\mathscr{G}$ as a monoidal category (as indeed Joyal has done [19] with second-player strategies alone) but with first-player strategies also, then a bifunctor $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ will not be able to represent the canonical addition $+$ since, for instance, we have $0 \twoheadrightarrow *$ but not $0 + 0 \twoheadrightarrow * + *$, which would follow from the assumption that $+$ is a functor from $A \times A$ to A.

We address this problem by introducing an alternative module structure on the product category $A \times B$.

Definition 3.1.7. Let A, B be game categories. The *exclusive-or product*, or *xor-product* for short, is the category A xor B with objects and normal arrows as in $A \times B$, but with module arrows as follows.

- If $f\colon x_1 \to y_1$ and $g\colon x_2 \twoheadrightarrow y_2$ then the pair $(f, g)\colon (x_1, x_2) \twoheadrightarrow (y_1, y_2)$.
- If $f\colon x_1 \twoheadrightarrow y_1$ and $g\colon x_2 \to y_2$ then the pair $(f, g)\colon (x_1, x_2) \twoheadrightarrow (y_1, y_2)$.

The *or-product* or $\vee$-product for short, denoted $A \vee B$, has objects and arrows as in $A \times B$ and A xor B, but includes all module arrows from both $A \times B$ and A xor B.

In each product the composition of module arrows and standard arrows is defined pointwise, in the obvious way (see Proposition 3.1.8 below).

Proposition 3.1.8. Each of the above notions of product forms a game category.

Proof. Assume A, B are game categories. We know that $A \times B$ is a category, and so it remains to show that the additional module structure is compatible in each case. It should be clear that we never have $x \nrightarrow x$. Suppose that

$$w \xrightarrow{f} u \overset{g}{\nrightarrow} v \xrightarrow{h} x \tag{3.1}$$

in $A \times B$, where each object and arrow is a pair consisting of something from A and something from B (so, for instance, $w = (w^A, w^B)$ and $f = (f^A, f^B)$). Then we have

$$\begin{aligned} w^A &\xrightarrow{f^A} u^A \overset{g^A}{\nrightarrow} v^A \xrightarrow{h^A} x^A; \\ w^B &\xrightarrow{f^B} u^B \overset{g^B}{\nrightarrow} v^B \xrightarrow{h^B} x^B. \end{aligned}$$

Hence the only sensible definition for $h \circ g \circ f$ is to take the pair $(h^A \circ g^A \circ f^A,\ h^B \circ g^B \circ f^B)$. Since this composition is associative in each component, it too is associative.

In the case of A xor B, if equation 3.1 holds then we have either

$$\begin{aligned} w^A &\xrightarrow{f^A} u^A \xrightarrow{h^A} v^A \xrightarrow{h^A} x^A; \\ w^B &\xrightarrow{f^B} u^B \overset{h^B}{\nrightarrow} v^B \xrightarrow{h^B} x^B; \end{aligned}$$

or the variant where h^A is a module arrow and h^B normal. In the former case, the obvious definition takes $h^A \circ h^A \circ f^A$ as the (normal) arrow in A, and $h^B \circ h^B \circ f^B$ as the (module) arrow in B. Analogously we define the composite for the case where h^A is the module arrow. Again, since this operation is component-wise associative, A xor B forms a game category.

Finally, in the case of $A \vee B$, we already have enough information to demonstrate that the module is compatible, since the two cases inherited from $A \times B$ and A xor B are disjoint. $\square$

These three constructions easily generalise to products of arbitrary classes $\mathscr{A} \subseteq \mathrm{GC}$. In particular $\mathrm{GC}^{\mathrm{PRO}}$ has full products.

It is important to note that the or- and xor-products are not products in the category-theoretic sense: they do not in general even admit projections, as shown in the following example.

Example 3.1.9. Let $A = \{1_A, a\}$ and $B = \{1_B, b\}$ be two-element game categories, with game category structure as depicted in figure 3.4. The true product $A \times B$ is then the game category of four objects, with no non-identity arrows, while the or- and exclusive-or-products are equal to the game category depicted in figure 3.5. Notice that there is no possibility of projections from A xor B to A and B.

$$
\begin{tikzcd}
1_A \arrow[d, leftrightarrow] & & 1_B \arrow[d, leftrightarrow, "|" marking] \\
a & & b
\end{tikzcd}
$$

Figure 3.4: Two-element game categories.

$$
\begin{tikzcd}
 & (1_A, 1_B) \arrow[dl, leftrightarrow, "|" marking] \arrow[dr, leftrightarrow, "|" marking] \arrow[dd, leftrightarrow] & \\
(1_A, b) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow, "|" marking] & & (a, b) \arrow[dl, leftrightarrow, "|" marking] \\
 & (a, 1_B) &
\end{tikzcd}
$$

Figure 3.5: The product A xor B.

3.1.2 Amphifunctors

We now address the issue of generalising amphimorphisms from chapter 2. This is not quite so simply done as with promorphisms, and while we can give a useful notion of natural transformation, there is not yet any obvious choice of definition for the corresponding module transformation.

Definition 3.1.10. Let A, B be game categories, and suppose $F\colon A \to B$ is a functor of the underlying categories. Assume there is a map $\overleftarrow{F}$ such that whenever arrows

$$
w \xrightarrow{f} x \qquad\qquad y \xrightarrow{h} z
$$

$$
\begin{tikzcd}
Fx \arrow[r, "g", "|" marking] & Fy
\end{tikzcd}
$$

exist in A and B, $\overleftarrow{F}x, yg$ is an arrow $x \nrightarrow y$ in A satisfying

$$
\overleftarrow{F}x, y(Fh \circ g \circ Ff) = h \circ \overleftarrow{F}x, yg \circ f. \tag{3.2}
$$

Then we call $(F, \overleftarrow{F})$ an *amphifunctor*, *amphi-game functor*, or *amphimorphism*.

In practice we will sometimes denote both maps simply by F. Since the respective images are in separate categories and the domains distinct, this should not cause confusion. We will also occasionally omit the subscript x, y when x and y are clear from the context. Notice that in particular if A, B are two-ordered structures then the amphifunctors $A \to B$ are precisely the amphimorphisms from A to B.

Proposition 3.1.11. The collection $\mathrm{GC}^{\mathrm{AM}}$, with object class GC and amphifunctors as appropriate arrows, forms a category.

Proof. We must prove that the composition of two amphifunctors is also an amphifunctor, and that this composition is associative. Assume

$$
A \xrightarrow{F} B \xrightarrow{G} C
$$

in $\mathrm{GC}^{\mathrm{AM}}$. Take GF to be the usual composition of the underlying functors. Define $\overleftarrow{GF}$ by

$$\overleftarrow{GF}x, y = \overleftarrow{F}x, y(\overleftarrow{G}Fx, Fyg)$$

for $g\colon GFx \nrightarrow GFy$. If we have $w, x, y, z \in A$ satisfying

$$w \xrightarrow{f} x \qquad\qquad y \xrightarrow{h} z$$

$$GFx \overset{g}{\nrightarrow} GFy$$

then

$$\overleftarrow{G}w, z(GFh \circ g \circ GFf) = Fh \circ \overleftarrow{G}x, yg \circ Ff,$$

so that

$$\begin{aligned}
\overleftarrow{GF}w, z(GFh \circ g \circ GFf) &= \overleftarrow{F}w, z(Fh \circ \overleftarrow{G}x, yg \circ Ff) \\
&= h \circ \overleftarrow{F}x, y \circ \overleftarrow{G}x, yg \circ f \\
&= h \circ \overleftarrow{GF}x, yg \circ f.
\end{aligned}$$

Therefore GF is an amphifunctor.

Now suppose

$$A \xrightarrow{F} B \xrightarrow{G} C \xrightarrow{H} D$$

in $\mathrm{GC}^{\mathrm{AM}}$. We know that the underlying functor composition is associative. Further, for $x, y \in A$,

$$\begin{aligned}
(\overleftarrow{(HG)F})x, y &= \overleftarrow{F}x, y \circ (\overleftarrow{HG})Fx, Fy \\
&= \overleftarrow{F}x, y \circ \left(\overleftarrow{G}Fx, Fy \circ \overleftarrow{H}GFx, GFy\right) \\
&= \left(\overleftarrow{F}x, y \circ \overleftarrow{G}Fx, Fy\right) \circ \overleftarrow{H}GFx, GFy \\
&= \overleftarrow{GF}x, y \circ \overleftarrow{H}GFx, GFy \\
&= \overleftarrow{H(GF)}x, y.
\end{aligned}$$

Therefore the composition of amphifunctors is associative. $\square$

Henceforth we shall not write the subscript x, y for the map $\overleftarrow{F}$. Otherwise the notation can be cumbersome, and in all cases it should be clear form the context which objects x and y are intended.

Since $\mathrm{GC}^{\mathrm{AM}}$ is a category, we should expect that $\mathrm{GC}^{\mathrm{AM}}[A, B]$ is a category for all objects A, B in $\mathrm{GC}^{\mathrm{AM}}$. With this in mind we make the following definition of appropriate natural transformation.

Definition 3.1.12. Suppose $F, G\colon A \to G$ in $\mathrm{GC}^{\mathrm{AM}}$, and that $\tau\colon F \to G$ is a natural transformation of the underlying functors. Assume also that whenever $g\colon Gx \nrightarrow Gy$ in $\operatorname{im} G = G[A]$, there is an arrow $\overleftarrow{\tau}x, y(g)\colon Fx \nrightarrow Fy$, satisfying

$$\overleftarrow{G}x, y(g) = \overleftarrow{F}x, y(\overleftarrow{\tau}x, y(g)).$$

Then τ is called a *natural transformation of amphifunctors*, or an *amphitransformation*.

Since this definition is much less conventional than the analogue for promorphisms, some explanation is required. Consider τ as above in terms of advantage for the second player. If II has a strategy in the image $\operatorname{im} F = F[A]$ of F, τ allows him to transfer this strategy to $G[A]$, in such a way that, regardless of when (i.e. for which game x) the transfer takes place from Fx to Gx, composition of strategies will always be the same. Dually, if I has a strategy in the image $G[A]$, then he has a strategy in $F[A]$ which corresponds to the same root strategy in A; that is, τ ensures that Player I will fare no better playing in $G[A]$ than in $F[A]$.

The following is easily proved.

Proposition 3.1.13. Let A, B be game categories. Then $\mathrm{GC}^{\mathrm{AM}}[A, B]$, the collection of amphifunctors $A \to B$, is a category with amphitransformations as arrows.

Products in $\mathbf{GC}^{\mathbf{am}}$

Definition 3.1.14. Let $\mathscr{A}$ be a class of game categories. We define a product $\prod^{\mathrm{AM}}\mathscr{A}$ as follows. Take

- $\mathrm{Obj}(\prod^{\mathrm{AM}}\mathscr{A}) = \prod \mathscr{A}$;
- $\mathrm{Arr}(\prod^{\mathrm{AM}}\mathscr{A}) = \prod_{A\in\mathscr{A}} \mathrm{Arr}(A)$;
- $\mathrm{MArr}(\prod^{\mathrm{AM}}\mathscr{A}) = \sum_{A\in\mathscr{A}} \mathrm{MArr}(A)$,

where $\sum_{A\in\mathscr{A}} \mathrm{MArr}(A)$ denotes the disjoint union of modules. The composition of normal arrows is defined in the obvious way, with $(g \circ f)(A) = g(A) \circ f(A)$. If $g\colon s \nrightarrow t$ in $\prod^{\mathrm{AM}}\mathscr{A}$ then g corresponds to a (unique) arrow $g_A\colon s(A) \nrightarrow t(A)$ in some A, and we define $f \circ g = f(A) \circ g_A$, $g_A \circ f = g \circ f(A)$.

In practice we will avoid writing g_A to distinguish the module arrow in A from that in $\prod^{\mathrm{AM}}\mathscr{A}$.

Suppose P denotes the projection in GC from $\prod^{\mathrm{AM}}\mathscr{A}$ to some $A \in \mathscr{A}$. If $g\colon a \nrightarrow b$ in A, then whenever $x, y \in \mathscr{A}$ with $x(A) = a$ and $y(A) = b$, we have $g\colon x \nrightarrow y$ in $\prod^{\mathrm{AM}}\mathscr{A}$; therefore we can set $\overleftarrow{P}x, y(g)$ as the arrow g in $\prod^{\mathrm{AM}}\mathscr{A}$. It is easily checked that this makes P into an amphimorphism.

Proposition 3.1.15. If $\mathscr{A} \subseteq \mathrm{GC}$ then $\prod^{\mathrm{AM}}\mathscr{A}$ is the product of $\mathscr{A}$ in $\mathrm{GC}^{\mathrm{AM}}$.

Proof. Assume $F_A\colon S \to A$ for all $A \in \mathscr{A}$, and let $M\colon S \to \prod^{\mathrm{AM}}\mathscr{A}$ be the mediating arrow for the product in the underlying category GC; that is, for $s \in S$, $Ms(A) = F_A s$, and for $f\colon s \to t$ in S, $Mf(A) = F_A f$. We show M can be equipped with a map $\overleftarrow{M}$ such that $(M, \overleftarrow{M})$ is an amphifunctor which acts as a mediator in $\mathrm{GC}^{\mathrm{AM}}$.

If $g\colon Ms_1 \nrightarrow Ms_2$ in $\prod^{\mathrm{AM}}\mathscr{A}$ then g is an arrow $Ms_1(A) \nrightarrow Ms_2(A)$ in some (unique) $A \in \mathscr{A}$; therefore $g\colon F_A s_1 \nrightarrow F_A s_2$, and we can define $\overleftarrow{M}g = \overleftarrow{F}Ag\colon s_1 \nrightarrow s_2$. To see this

makes M into an amphifunctor, suppose $f\colon s_0 \to s_1$ and $h\colon s_2 \to s_3$ in S. Then

$$\begin{aligned}
\overleftarrow{M}(Mh \circ g \circ Mf) &= \overleftarrow{M}(F_A h \circ g \circ F_A f) \\
&= \overleftarrow{F}_A(F_A h \circ g \circ F_A f) \\
&= h \circ \overleftarrow{F}_A g \circ f \\
&= h \circ \overleftarrow{M} g \circ f,
\end{aligned}$$

and so M is an amphifunctor.

We now show that for each projection $P_A\colon \prod^{\mathrm{AM}} \mathscr{A} \to A$, we have $P_A \circ M = F_A$. This is already evident for the underlying functors. If $g\colon F_A s \nrightarrow F_A t$ in A then

$$\begin{aligned}
\overleftarrow{P_A \circ M} g &= \overleftarrow{M}(\overleftarrow{P}_A g) \\
&= \overleftarrow{M} g \\
&= \overleftarrow{F}_A g;
\end{aligned}$$

hence $\overleftarrow{P_A \circ M} = \overleftarrow{F}_A$.

Finally if some amphifunctor $N\colon S \to \prod^{\mathrm{AM}} \mathscr{A}$ also satisfies this property then the object and normal arrow assignment of N must be equal to that of M; further, if $g\colon Ns_1 \nrightarrow Ns_2$ in $\prod^{\mathrm{AM}} \mathscr{A}$, $g\colon F_A s_1 \nrightarrow F_A s_2$ in some A; so $\overleftarrow{N} g = \overleftarrow{N}(\overleftarrow{P}_A g) = \overleftarrow{F}_A g = \overleftarrow{M} g$, so that $N = M$. □

3.2 Enriched game categories and the value map

Briefly we concern ourselves with enriched game categories. Such discussion will be of use here and in section 3.4. To clarify notation we recall the definition of an enriched category.

Definition 3.2.1. Assume $(M, \otimes, e_M)$ is a monoidal category. An M-enriched category, or a category enriched over M, is a tuple

$$A = (\mathrm{Obj}(A), A[-,-], \mathrm{id}, \circ),$$

where

- $\mathrm{Obj}(A)$ is a class of objects;
- $A[-,-]$ is a function $\mathrm{Obj}(A) \times \mathrm{Obj}(A) \to M$;
- for all $a \in A$, id_a is an M-morphism $e^M \to A[a,a]$;
- $\circ$ is a partial function on $A \times A \times A$ such that for all $a, b, c \in A$,

$$\circ_{a,b,c}\colon A[b,c] \otimes A[a,b] \to A[a,c]$$

  is a morphism in M;

and furthermore the diagrams in figures 3.6 and 3.7 commute.

$$\begin{tikzcd}
(A[c,d] \otimes A[b,c]) \otimes A[a,b] \arrow[rr, "\alpha^M"] \arrow[d, "\circ\otimes 1"'] & & A[c,d] \otimes (A[b,c] \otimes A[a,b]) \arrow[d, "1\otimes\circ"] \\
A[b,d] \otimes A[a,b] \arrow[dr, "\circ"'] & & A[c,d] \otimes A[a,c] \arrow[dl, "\circ"] \\
& A[a,d] &
\end{tikzcd}$$

Figure 3.6: Associativity of composition.

$$\begin{tikzcd}
e^M \otimes A[a,b] \arrow[dr, "\lambda"] \arrow[dd, "\mathrm{id}_a \otimes 1"'] & & A[a,b] \otimes e^M \arrow[dl, "\rho"'] \arrow[dd, "1\otimes\mathrm{id}_a"] \\
& A[a,b] & \\
A[a,a] \otimes A[a,b] \arrow[ur, "\circ"'] & & A[a,b] \otimes A[a,a] \arrow[ul, "\circ"]
\end{tikzcd}$$

Figure 3.7: Left and right-identity rules of composition.

Extending this definition to game categories is fairly straightforward. Suppose we have an additional function, $A(-,-)\colon A \times A \to M$, characterising the first-player strategies in A. In order to simulate the axioms of definition 3.1.1 we require that the composition map is also defined for all pairs of the form $(A[b,c], A(a,b)), (A(b,c), A[a,b])$, and that the diagrams in figures 3.8, 3.9 and 3.10 commute.

$$\begin{tikzcd}
(A[c,d] \otimes A(b,c)) \otimes A[a,b] \arrow[rr, "\alpha"] \arrow[d, "\circ\otimes 1"'] & & A[c,d] \otimes (A(b,c) \otimes A[a,b]) \arrow[d, "1\otimes\circ"] \\
A(b,d) \otimes A[a,b] \arrow[dr, "\circ"'] & & A[c,d] \otimes A(a,c) \arrow[dl, "\circ"] \\
& A(a,d) &
\end{tikzcd}$$

Figure 3.8: Central associativity for mixed composition.

To simulate the axiom $\forall x\, (x \not\twoheadrightarrow y)$, we can require that for each $x \in A$, $A(x,x)$ is initial; in the case where M is a category of sets (or the category $2 = \{0 \to 1\}$), this is equivalent to the statement that no module arrow $a \twoheadrightarrow a$ exists, for all $a \in A$. If $A = (\mathrm{Obj}(A), A[-,-], A(-,-), \mathrm{id}, \circ)$ satisfies these criteria then we call A an *M-enriched game category*, or a *game category* over M.

$$
\begin{tikzcd}
(A[c,d] \otimes A[b,c]) \otimes A(a,b) \arrow[rr, "\alpha"] \arrow[d, "\circ\otimes 1"'] & & A[c,d] \otimes (A[b,c] \otimes A(a,b)) \arrow[d, "1\otimes \mathrm{id}_a"] \\
A[b,d] \otimes A(a,b) \arrow[dr, "\circ"'] & & A[c,d] \otimes A(a,c) \arrow[dl, "\circ"] \\
& A(a,d) &
\end{tikzcd}
$$

Figure 3.9: Left associativity for mixed composition.

$$
\begin{tikzcd}
(A(c,d) \otimes A[b,c]) \otimes A[a,b] \arrow[rr, "\alpha"] \arrow[d, "\circ\otimes 1"'] & & A(c,d) \otimes (A[b,c] \otimes A[a,b]) \arrow[d, "1\otimes \mathrm{id}_a"] \\
A(b,d) \otimes A[a,b] \arrow[dr, "\circ"'] & & A(c,d) \otimes A[a,c] \arrow[dl, "\circ"] \\
& A(a,d) &
\end{tikzcd}
$$

Figure 3.10: Right associativity for mixed composition.

Of particular interest is enrichment over 2, the category with non-identity arrow $0 \to 1$. If A is any game category then following Joyal [19] and Conway et al. [7, 4] we define relations $\leq, \lhd|$ by $x \leq y$ if and only if $A[x,y] \neq \varnothing$ and $x \lhd| y$ if and only if $A(x,y) \neq \varnothing$. In particular if A is a game category over 2, then the two-ordered structure thusly obtained is essentially the same as the category-theoretic structure of A. We obtain a map $\mathrm{T}_{\leq,\lhd|}\colon \mathrm{GC}^{\mathrm{PRO}} \to \textsc{Tos}$. Moreover there is an obvious embedding functor, $\mathrm{Emb}\colon \textsc{Tos} \to \mathrm{GC}$, for which we can easily prove the following.

Proposition 3.2.2. The map $\mathrm{T}_{\leq,\lhd|}\colon \mathrm{GC}^{\mathrm{PRO}} \to \textsc{Tos}$ is a functor, and is left-adjoint to Emb.

By $\textsc{Pre}, \textsc{Pos}$ we denote the categories of preordered and partially ordered classes respectively. If $(P, \leq) \in \textsc{Pre}$ then we can define $\simeq$ on P as usual (that is, $x \simeq y$ if and only if $x \leq y \leq x$). We define $\mathrm{Q}_{\simeq}(P)$ to be the quotient $(P/{\simeq}) \in \textsc{Pos}$. The functor $\mathrm{Q}_{\simeq}$ is left-adjoint to the embedding functor $\mathrm{Emb}_{\simeq}\colon \textsc{Pos} \to \textsc{Pre}$. We can easily extend this construction to two-ordered structures: if $(S, \leq, \lhd|) \in \textsc{Tos}$ then $\mathrm{Q}_{\simeq}(S)$ is easily made into a two-ordered structure by setting $(x/{\simeq}) \lhd| (y/{\simeq})$ if and only if $x \lhd| y$. Clearly this is well-defined, and further $\mathrm{Q}_{\simeq}(S)$ is an exact two-ordered structure.

Proposition 3.2.3. Let $\mathrm{E}\textsc{Tos}$ denote the category of exact two-ordered structures. Then $\mathrm{Q}_{\simeq}\colon \textsc{Tos} \to \mathrm{E}\textsc{Tos}$ is a functor, and is left-adjoint to $\mathrm{Emb}_{\simeq}\colon \mathrm{E}\textsc{Tos} \to \textsc{Tos}$.

In ONAG and Winning Ways the value of a game x is vaguely associated with the equivalence class of x modulo $\simeq$. Generalising this we define the *value space* of a game category A to be

$$\mathrm{Val}(A) = \mathrm{Q}_{\simeq}(\mathrm{T}_{\leq,\lhd|}(A)).$$

Letting $\mathrm{E} = \mathrm{Emb}_{\simeq} \circ \mathrm{Emb}$, we obtain the following.

Proposition 3.2.4. The *value map* Val: $\mathrm{GC}^{\text{PRO}} \to \mathrm{ET}\textsc{os}$ is a functor, and is left-adjoint to the embedding functor E: $\mathrm{ET}\textsc{os} \to \mathrm{GC}^{\text{PRO}}$.

Although not particularly inspiring this result does help to explain why so much of the theory of Conway games can be deduced by considering only games' values. Later on we will see that the adjunction is preserved when we add extra structure to our game categories, giving a less trivial result.

Remarks 3.2.5.

- For games x in any game category A, let val(x) denote the quotient $x/\simeq$ in Val(A) (the *value of* x). Then the map val: $A \to \mathrm{Val}(A)$ is easily shown to be a promorphism. We cannot make this map into an amphimorphism, however, since in general there will be no canonical choice of first-player strategies in A.
- Notice that for a two-ordered structure S, S is exact if and only if $\mathrm{Aut}\, S$ is exact. Therefore by theorems 2.4.2 and 2.4.4, exact two-ordered groups are precisely the automorphism groups of games' values.

3.3 Monoidal game categories

Now we look at game categories with additional monoidal structure. Recall that Joyal [19] showed the collection of wellfounded partisan games from ONAG and Winning Ways forms a compact closed monoidal category; we extend this to include first-player strategies. Our choice of functor (amphi- or pro-) greatly affects the behaviour of any monoidal product, and in particular amphifunctors introduce a new level of complexity. For simplicity we will concern ourselves only with promorphisms, from now on referred to simply as *game functors*. We can view these functors, and the transformations between them, as the 1-cells and 2-cells in the category GC^{PRO}.

Definition 3.3.1. Assume M is a game category, and $\otimes\colon M \text{ xor } M \to M$ is a game functor. Suppose that

$$\begin{aligned} \alpha_{x,y,z}&\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z), \\ \lambda_x&\colon e^M \otimes x \to x, \\ \rho_x&\colon x \otimes e^M \to x \end{aligned}$$

are isomorphisms, natural in x, y, z (that is, pro-transformations which are also isomorphisms), such that the underlying category structure of $(M, \otimes, e^M, \alpha, \lambda, \rho)$ is a monoidal category (cf. Mac Lane [25, p.162]). Then $(M, \otimes, e^M, \alpha, \lambda, \rho)$ is a *monoidal game category*.

We have already discussed the justification for requiring that $\otimes$ be a functor with domain M xor M: this ensures that whenever $f\colon x_0 \to y_0$ and $g\colon x_1 \twoheadrightarrow y_1$, both $f \otimes g\colon x_0 \otimes x_1 \twoheadrightarrow y_0 \otimes y_1$ and $g \otimes f\colon x_1 \otimes x_0 \twoheadrightarrow y_1 \otimes y_0$. This is true of, for example, the disjunctive sum under the normal play condition.

Definition 3.3.2. Assume M, N are monoidal game categories. A monoidal game functor $F\colon M \to N$ is a tuple (F_0, μ^F, ι^F) such that

- $F_0\colon M \to N$ is a game functor;
- (F_0, μ^F, ι^F) is a strong monoidal functor in the sense of Mac Lane [25, p.255], ignoring the module structure.

If $F, G\colon M \to N$ are monoidal game functors then a monoidal (game) transformation $\tau\colon F \to G$ is simply a monoidal transformation (see Mac Lane [25, p.256]) of the underlying functors which also makes the diagram in figure 3.11 commute for all $x, y \in M$.

$$\begin{tikzcd} Fx \otimes Fy \arrow[r, "\mu^F_{x,y}"] \arrow[d, "\tau_x \otimes \tau_y"'] & F(x \otimes y) \arrow[d, "\tau_{x\otimes y}"] \\ Gx \otimes Gy \arrow[r, "\mu^G_{x,y}"'] & G(x \otimes y) \end{tikzcd}$$

Figure 3.11: Monoidal natural transformations of game functors.

By MGC we denote the collection of monoidal game categories with monoidal game functors as arrows. This can be seen as a 2-category, where the module 2-cells of MGC are precisely the module 2-cells of $\mathrm{GC}^{\mathrm{PRO}}$. With these transformations of functors, the following is easily shown.

Proposition 3.3.3. If A is a game category then $\mathrm{End}(A) = \mathrm{GC}^{\mathrm{PRO}}[A, A]$ is a (strict) monoidal game category.

3.3.1 Monoidal game categories and the value map

Recall that the value map $\mathrm{Val}\colon \mathrm{GC}^{\mathrm{PRO}} \to \mathrm{ETos}$ is a game functor and part of an adjunction. It is natural to consider the restriction of this map to MGC, $\mathrm{Val}\restriction \mathrm{MGC}$. If $M \in \mathrm{MGC}$, however, $\mathrm{Val}(M)$ does not necessarily have compatible monoidal structure. We saw previously that Val is the composition of functors

$$\mathrm{GC}^{\mathrm{PRO}} \overset{\mathrm{T}_{\leq,\lhd\!\shortmid}}{\longrightarrow} \mathrm{Tos} \overset{\mathrm{Q}_{\simeq}}{\longrightarrow} \mathrm{ETos},$$

and in fact $\mathrm{T}_{\leq,\lhd\!\shortmid}(M)$ does have suitable structure for a tensor product. This is because the class $\mathrm{Obj}(M)$ remains the same, and so we can merely keep the product $\otimes^M$ unchanged in $\mathrm{T}_{\leq,\lhd\!\shortmid}(M)$. That is, $(\mathrm{T}_{\leq,\lhd\!\shortmid}(M), \otimes^M)$ is a monoid, and further $(\mathrm{T}_{\leq,\lhd\!\shortmid}(M), \otimes(M))$ satisfies axioms T1-T4 of definition 2.1.1 (therefore $\mathrm{T}_{\leq,\lhd\!\shortmid}(M)$ is a *two-ordered monoid*). In particular, $\mathrm{T}_{\leq,\lhd\!\shortmid}\restriction \mathrm{MGC}$ may be considered a monoidal game functor when equipped with this extra information (this is trivial, since every arrow in $\mathrm{T}_{\leq,\lhd\!\shortmid}(M)$ is reduced to a single relation between objects).

It is the quotient functor $\mathrm{Q}_{\simeq}$ which fails to preserve the multiplication, since for all $x, y \in M$, $z \simeq x \otimes y$ does not necessarily imply $z = x' \otimes y'$ for some $x' \simeq x$ and $y' \simeq y$.

This can be avoided by requiring that M is *rigid*. Recall that a pair of dual objects in a monoidal category $(M, \otimes, e)$ is a pair (x, y) with morphisms

$$\eta\colon 1 \to y \otimes x,$$
$$\varepsilon\colon x \otimes y \to 1,$$

such that

$$\lambda_x \circ (\varepsilon \otimes \mathrm{id}_x) \circ \alpha_{x,y,x}^{-1} \circ (\mathrm{id}_x \otimes \eta) \otimes \rho_x^{-1} = \mathrm{id}_x;$$
$$\rho_y \circ (\mathrm{id}_y \otimes \varepsilon) \circ \alpha_{y,x,y} \circ (\eta \otimes \mathrm{id}_y) \circ \lambda_y^{-1} = \mathrm{id}_y .$$

Definition 3.3.4. Let M be a monoidal game category. We call M a *rigid game category* if for each $x \in M$ there is $y \in M$ such that (x, y) is a dual pair.

It is easily seen that in the value space of a rigid monoidal category, the product is compatible with the relations $\leq$ and $\lhd\!\shortmid$.

Proposition 3.3.5. If M is a rigid game category then the product $\otimes$ on $\mathrm{Val}(M)$ is well-defined, and makes $(\mathrm{Val}(M), \leq, \lhd\!\shortmid)$ into a two-ordered group. Moreover, $\mathrm{val}\colon M \to \mathrm{Val}(M)$ is a monoidal game functor.

Example 3.3.6. Let $\mathscr{G}$ denote a class of partisan games (cf. ONAG [7], Winning Ways [4]) containing $0 = \{\,|\,\}$, and suppose $\mathscr{G}$ is closed under addition and negation. Then $\mathscr{G}$ is a (strict) rigid game category, where $(x, -x)$ is a dual pair for each x. The strategies η and ε in this case are different realisations of the copycat strategy $1_x\colon x \to x$, as arrows $0 \to -x + x$ and $x - x \to 0$.

3.4 Architectures

In any theory of games a notion of membership can help to connect the notions of strategy, option, position and game. In particular viewing each game as a two-sided container, whose elements are respective players' options, allows us to view these as equivalent notions. We formulate the following definition based primarily on discussion by Joyal [19], and Cockett et al. [6].

Definition 3.4.1. Assume $(V, \otimes, e^V, \ldots)$ is a monoidal category with full products and coproducts. Let A be a game category enriched over V with binary relations $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$. We say A is *instructive* if for all $x, y \in A$ there are S-arrows

$$\sigma_0(x,y)\colon A[x,y] \longrightarrow \prod_{u \in_{\mathrm{L}} x} A(u,y) \times \prod_{v \in_{\mathrm{R}} y} A(x,v), \tag{3.3}$$

$$\sigma_1(x,y)\colon A(x,y) \longrightarrow \sum_{u \in_{\mathrm{R}} x} A[u,y] \uplus \sum_{v \in_{\mathrm{L}} y} A[x,v], \tag{3.4}$$

natural in x, y.

Dually we call $(A, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ *constructive* if for all $x, y \in A$ there are Set-arrows

$$\tau_0(x,y)\colon A[x,y] \longleftarrow \prod_{u \in_{\mathrm{L}} x} A(u,y) \times \prod_{v \in_{\mathrm{R}} y} A(x,v), \tag{3.5}$$

$$\tau_1(x,y)\colon A(x,y) \longleftarrow \sum_{u \in_{\mathrm{R}} x} A[u,y] \uplus \sum_{v \in_{\mathrm{L}} y} A[x,v], \tag{3.6}$$

natural in x, y.

If A is both instructive and constructive, and further if $\tau(x,y), \sigma(x,y)$ form a pair of mutual inverses for each x, y, then we call A (or more properly, $(A, \in_{\mathrm{L}}, \in_{\mathrm{R}}, \tau_0, \ldots)$) an *architecture* over S or an S*-architecture*.

The instructive property ensures that, given a second player strategy $f \in A[x,y]$, we may find first-player strategies regardless of Right's move. Dually, given a first-player strategy $g \in A(x,y)$ we may find a second-player strategy in at least one Left option. The constructive property allows us to build new strategies in more complex games, given an appropriate collection of strategies for their options. Notice that an architecture does not require monoidal structure; indeed, there may be a method for combining games which is not monoidal, or we may not have any such structure.

Architectures compare closely with the combinatorial game categories of Cockett et al. [6], since each notion is essentially that of a module category carrying additional set-theoretic structure designed to capture moves within games. Ignoring our requirement that $a \not\twoheadrightarrow a$ for all games a, the most significant difference is that architectures lack the closure under finite diproducts that cgcs enjoy. This in itself allows architectures to model classes of games which do not form cgcs themselves (though, if enriched over an appropriate category an architecture will generate a cgc).

Example 3.4.2. In many versions of Hackenbush there is certainly no obvious definition of diproduct. In, for example, restrained hackenbush [7, p.86] it is impossible to define a diproduct since no such game is confused with 0. If this collection were a combinatorial game category then a game of the form $\{0 \,|\, 0\}$, which is confused with 0, would exist.

Question 3.4.3. Which versions of Hackenbush have a compatible diproduct?

As another example, consider the set $\{P, N, L, R\}$ (analogous to the set $\{0, *, 1, -1\}$ of Conway games, in that order), with arrows $x \to y$ when $x \leq y$ (in Conway's sense) and $x \twoheadrightarrow y$ when $x \lhd\!\mid y$, discussed by Cockett et al. [6, Example 5.5, p.18]. As a result of the diproduct, projection and injection rules, there are additional arrows which do not appear in the typical structure $\{0, *, -1, 1\}$: for instance the diproduct $\{P|L\}$ is equal to L, and so by injection $L \twoheadrightarrow L$. Thus the insistence of closure under diproducts introduces unwanted arrows. This structure is better represented as an architecture.

Although it may seem a trivial extra condition, our insistence that $a \not\twoheadrightarrow a$ for all games a in the architecture A also imposes a weak form of regularity in the set-theoretic structure. For instance, we cannot have that a is an element of itself, although non-wellfounded architectures with loops such as $a \in_{\mathrm{L}} b \in_{\mathrm{R}} a$ exist. It is, in fact, impossible to have $a \in_{\mathrm{P}} a$ whenever A is either instructive or constructive, as a consequence of the axiom $\forall a\, (a \not\twoheadrightarrow a)$.

3.4.1 Architectures and the value map

Briefly we consider architectures and the value map of section 3.2. If A is an architecture then the most reasonable definition of membership within the value space $\mathrm{Val}(A)$ is

$$\mathrm{val}(u) \in_{\mathrm{P}} \mathrm{val}(x) \Leftrightarrow \exists u' \simeq u\ \exists x' \simeq x\,(u \in_{\mathrm{P}} x).$$

This definition is in part a mathematical convenience; however it is in keeping with the view of games as Dedekind cuts.

Proposition 3.4.4. If $(A, \ldots)$ is an architecture then $\mathrm{Val}(A)$, with the usual relations and above-defined memberships, is an architecture over 2.

Let Arch denote the collection of architectures. A functor from the U-architecture A to the V-architecture B is best defined to be an enriched functor of the underlying categories which also preserves both the module structure (making it a game functor) and the membership relations. From the game-theoretic perspective this is sensible: such a map preserves playability. These functors make the collection Arch into a category. It is easily seen that the assignment of values on an architecture is such a functor. That is, for $A \in \mathrm{Arch}$, $\mathrm{val}\colon A \to \mathrm{Val}(A)$ is an architecture functor. As above, we can also show that the map Val, which takes a game category to its value space, is part of an adjunction.

3.5 Gamuts

In this section we show that the wellfounded games of ONAG and Winning Ways are examples of the structures discussed above. Notice in particular that this applies, not just to the "pure" partisan games (that is, two-sided containers which behave like sets; see chapter 4), but also to the distinct games, such as Hackenbush, Domineering, Go.

Definition 3.5.1. Suppose G is a monoidal game category enriched over the monoidal category V. If $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ are binary relations on G, and V has natural isomorphisms τ_0, τ_1 which also make $(G, \in_{\mathrm{L}}, \in_{\mathrm{R}}, \ldots)$ an architecture, then G is called a *gamut*.

We define gamut functors similarly, as maps with monoidal and set-theoretic structure.

Definition 3.5.2. Suppose G, H are gamuts, and that $F\colon G \to H$ is an architecture functor. If there exist natural transformations ι^F and μ_F such that (F, μ^F, ι^F) is a monoidal game functor, then F is a *gamut functor*.

By Gam we denote the category of gamuts and their functors.

3.5.1 Examples of gamuts

We now argue, with informal proofs, that the wellfounded games of ONAG [7] and Winning Ways [4] exhibit the properties discussed above. This discussion includes specific classes of games (for instance Hackenbush, Domineering and Go), as well as the "pure"

partisan games (that is, two-sided containers which behave like sets; see chapter 4). Moreover the results which follow are independent of any representation. For example, Hackenbush games can be realised as (for instance) trees, tuples of positions or intuitively as diagrams on the plane; each representation is valid. For clarity we make the following definitions.

Definition 3.5.3. Let A be an architecture. For $x, y \in A$ we write $x \in^{\mathrm{L}}_{\mathrm{R}} y$ if and only if $x \in_{\mathrm{L}} y$ or $x \in_{\mathrm{R}} y$. We call A *wellfounded* if and only if the relation $\in^{\mathrm{L}}_{\mathrm{R}}$ is wellfounded, i.e.

$$\forall x\big(\exists y(y \in^{\mathrm{L}}_{\mathrm{R}} x) \to \exists y \in^{\mathrm{L}}_{\mathrm{R}} x \, \forall z \in^{\mathrm{L}}_{\mathrm{R}} x(z \notin^{\mathrm{L}}_{\mathrm{R}} y)\big).$$

Our claim can now be stated more concisely, as follows.

Theorem 3.5.4. Each wellfounded structure $\mathcal{H}$ of games considered in ONAG and Winning Ways is a gamut, where the disjunctive sum $+$ and negation $-$ determined the rigid monoidal structure.

Before we discuss the proof of this theorem it will be helpful to introduce some new notation and terminology. If $x \in_{\mathrm{P}} y$ where $\mathrm{P} \in \{\mathrm{L}, \mathrm{R}\}$, we will refer to x as a P-option of y, and as in the literature, we will use (for example) x^{L} to stand for an arbitrary element $u \in_{\mathrm{L}} x$. As in chapter 4 we denote by x^{P} an arbitrary P-option of x.

Suppose A is any instructive game category, in which $f \colon x \to y$. Then for any x^{L} and any y^{R} there exist strategies induced by f. Explicitly, since $f \in A[x, y]$ and

$$A[x, y] \cong \prod_{x^{\mathrm{L}}} A(x^{\mathrm{L}}, y) \times \prod_{y^{\mathrm{R}}} A(x, y^{\mathrm{R}}),$$

we can pick particular first-player strategies $f{\downarrow}x^{\mathrm{L}} \colon x^{\mathrm{L}} \twoheadrightarrow y$ and $f{\downarrow}y^{\mathrm{R}} \colon x \twoheadrightarrow y^{\mathrm{R}}$ for all x^{L} and y^{R}. In cases where ambiguity is likely (for instance, if some $u \in A$ satisfies $u \in_{\mathrm{L}} x$ and $u \in_{\mathrm{R}} y$), we write, for example, $f{\downarrow}(u \in_{\mathrm{L}} x)$ or $f{\downarrow}(u \in_{\mathrm{R}} y)$ to distinguish the separate meanings.

In the case of first-player strategies, we can also define an induced strategy. If $f \colon x \twoheadrightarrow y$, then since A is an architecture f corresponds to a second player strategy, either $x^{\mathrm{R}} \to y$ or $x \to y^{\mathrm{L}}$. Again we will write $f{\downarrow}x^{\mathrm{R}}$ or $f{\downarrow}y^{\mathrm{L}}$ respectively, and in cases where ambiguity is possible we instead write $f{\downarrow}(u \in_{\mathrm{R}} x)$ or $f{\downarrow}(u \in_{\mathrm{L}} x)$; notice, however, that in this case $f{\downarrow}$ is only a partial function, applying to a single argument.

In some cases – depending upon the particular definition of "strategy" in A – f will be a function, and we will have $f(x^{\mathrm{P}}) = f \downarrow x^{\mathrm{P}}$. However it is useful to fix additional notation for the next position advocated by f, in order to distinguish this position from the induced strategy. If f is a first-player strategy in x, say, then $f[x]$ will be the next position advocated by a first-player strategy f. Hence $f{\downarrow}f[x]$ determines the remaining play dictated by f.

When A is an architecture, we can define a new strategy $f \colon x \to y$ by first describing the strategies $f{\downarrow}x^{\mathrm{L}}$ and $f{\downarrow}y^{\mathrm{R}}$. Similarly we can describe $g \colon x \twoheadrightarrow y$ by defining any $g{\downarrow}x^{\mathrm{R}}$ or $g{\downarrow}y^{\mathrm{L}}$.

Henceforth, we fix a class $\mathcal{H}$ of games, and assume

- $\in^{\mathrm{L}}_{\mathrm{R}}$ is wellfounded on $\mathcal{H}$;

- $\mathcal{H}$ is closed under disjunctive sums and negation (however they may be defined);
- $\mathcal{H}$ contains a zero game.

The following is obvious. Indeed, if we work within a monoidal category $(\mathrm{Set}, \times)$ of sets then the instructive and constructive rules of definition 3.4.1 dictate an appropriate definition of strategies, which makes the isomorphisms τ_0, τ_1 identities.

Proposition 3.5.5. With the (usually obvious) definition of strategy, $\mathcal{H}$ is an architecture.

We can also show that the existence of a disjunctive sum leads to a monoidal product, as indicated by Joyal [19]. We assume the existence of a map $+\colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ (where here $\mathcal{H} \times \mathcal{H}$ denotes a product of underlying categories) satisfying

$$\forall x, y, z \ \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} x + y \ \leftrightarrow \exists w \in_{\mathrm{P}} x \,(z = w + y) \vee \exists w \in_{\mathrm{P}} y \,(z = x + w)\big).$$

That is, $x + y$ has the form $\{x^{\mathrm{P}} + y, x + y^{\mathrm{P}}\}_{\mathrm{P}}$, although $x + y$ may not be the unique such game: in many cases the relations $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ may not satisfy an appropriate extensionality axiom[1]. We also assume that a negation functor $-\colon \mathcal{H}^{\mathrm{op}} \to \mathcal{H}$, satisfying

$$\forall x \ \forall y \ \left(\bigwedge_{\mathrm{P}} (y \in_{\mathrm{P}} -x \leftrightarrow \exists z \in_{\neg\mathrm{P}} x \,(y = -z)\right),$$

exists; that is, $-x$ has the form $\{-y\colon y \in_{\mathrm{R}} x \mid -y\colon y \in_{\mathrm{L}} x\}$.

Proposition 3.5.6. Under these assumptions, $\mathcal{H}$ is a rigid game category.

Sketch proof. We merely provide appropriate definitions and some guidance here, as these ideas have long been understood. If $f\colon w \to x$ and $g\colon y \to z$ in A, we recursively define the sum by taking

$$\begin{aligned}
(f + g)\!\downarrow\!(w^{\mathrm{L}} + y) &= (f\!\downarrow\! w^{\mathrm{L}}) + g;\\
(f + g)\!\downarrow\!(w + y^{\mathrm{L}}) &= f + (g\!\downarrow\! y^{\mathrm{L}});\\
(f + g)\!\downarrow\!(x^{\mathrm{R}} + z) &= (f\!\downarrow\! x^{\mathrm{R}}) + g;\\
(f + g)\!\downarrow\!(x + z^{\mathrm{R}}) &= f + (g\!\downarrow\! z^{\mathrm{R}}).
\end{aligned}$$

If $f\colon w \to x$ and $g\colon y \nrightarrow z$, either some $g\!\downarrow\! y^{\mathrm{R}}$ exists, and so we define $(f + g)\!\downarrow\!(w + y^{\mathrm{R}})$ to be $f + (g\!\downarrow\! y^{\mathrm{R}})$; or some $g\!\downarrow\! z^{\mathrm{L}}$ exists, and we define $(f + g)\!\downarrow\! z^{\mathrm{L}}$ to be $f + (g\!\downarrow\! z^{\mathrm{L}})$. Notice that since A is an architecture, the strategy g corresponds to exactly one such possibility, so the sum strategy is well defined (and requires no element of choice). The treatment for $g + f$ is analogous.

[1]The obvious axiom here is

$$\forall x \ \forall y \ \left(\left(\bigwedge_{\mathrm{P}} \forall z \,(z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)\right) \to x = y\right);$$

see chapter 4.

In this way we define all sums $f+g$ of strategies f, g. It is now easily proved by induction (which is possible since $\in^{\mathrm{L}}_{\mathrm{R}}$ is wellfounded) that the function $+$, along with this arrow assignment, defines an mgc functor. $\square$

From Propositions 3.5.5 and 3.5.6 we deduce the following.

Theorem 3.5.7. The structure $\mathcal{H}$ is a gamut.

We can also demonstrate that every wellfounded gamut maps onto a gamut of pure games via a gamut functor, which is part of an adjunction. Such a statement deserves caution, however. We have avoided making reference to "the collection of partisan games", often denoted **Pg**, of ONAG [7, ch.7] and Winning Ways [4]. Rather, we consider any class of games which contains non-impartial games to be such a collection. Moreover, we take the view that any model of amphi-ZF (cf. chapter 4), or possibly of a weaker theory, is a viable candidate for **Pg**. While the distinction is unnecessary for the study of finite games, different models will produce important variations on the theory of infinite games. In the case of wellfounded gamuts (which also contain an empty object), we can state the following.

Proposition 3.5.8. For each wellfounded gamut $\mathcal{H}$ there is a wellfounded gamut of pure games, $\mathcal{H}$, and a full gamut functor $F\colon \mathcal{H} \to \mathcal{H}$.

In stating this result we have ignored particular foundational issues: in particular the proof of proposition 3.5.8 involves an assumption that we can factor $\mathcal{H}$ to obtain an extensional (but categorically equivalent) category $\mathcal{H}$ (such a quotient would likely be formed using Scott's trick). Depending on our definition of category we might also prove that $\mathcal{H}$ is equivalent to some model $\mathcal{H}$ of amphi-ZF, using a Mostowski collapse.

Proposition 3.5.9. Ignoring foundational issues and assuming global choice, we can prove the existence of an equivalence $F\colon \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a gamut of pure two-sided containers (i.e. a model of amphi-ZF).

We remark that such assumptions are consistent with those made earlier, while discussing the value map.

CHAPTER 4

AMPHI-ZF

4.1 Introduction

In this chapter we will introduce a theory of *amphisets*, called amphi-ZF and denoted ZF_2 or AZF. This is a generalisation of ZF to a theory of two-sided containers; the axioms we give will easily generalise to any number of inclusions $(\in_i)_{i\in I}$[1], however we are primarily interested in the case $I = 2$. Once this theory has been introduced we prove it to be synonymous with ZF; that is, the two theories may be considered the same, with each container in one theory corresponding to one in the other, and vice versa.

By *interpretation* we shall mean a *relative interpretation*, as defined by Visser [34]. We briefly describe such objects and their category-theoretic framework, which makes our discussion much simpler, here. The category INT has as objects logical theories. All theories are assumed to have only relations as non-logical symbols; among these we assume a unary relation δ, indicating the domain, is included (note that equality is included as a logical symbol). We assume full first-order logic, including the equality rules, and the logical axiom $\forall x\ \delta(x)$. By a *relative translation* $\mathfrak{f}\colon T_2 \to T_1$ of an $\mathscr{L}_2$-theory T_2 into an $\mathscr{L}_1$-theory T_1 we mean a mapping $\mathfrak{f}$ of atomic formulas $R(x_0,\ldots,x_{n-1})$ of $\mathscr{L}_2$ to formulas $R(x_0,\ldots,x_{n-1})^{\mathfrak{f}}$ of $\mathscr{L}_1$, in the same free variables. In particular $\delta(x)^{\mathfrak{f}}$ is some formula $\delta_{\mathfrak{f}}(x)$ specifying the domain, and $(x = y)^{\mathfrak{f}}$ is in our case simply $(x = y)$ (in Visser's terms we work within the subcategory $\mathrm{INT}_=$ of INT). This mapping is extended to all $\mathscr{L}_2$-formulas by taking $(\neg\,\theta(\overline{x}))^{\mathfrak{f}}$ to be $\neg\,\theta(\overline{x})^{\mathfrak{f}}$, $(\phi(\overline{x}) \to \psi(\overline{x}))^{\mathfrak{f}}$ to be $\phi(\overline{x})^{\mathfrak{f}} \to \psi(\overline{x})^{\mathfrak{f}}$, and $(\forall\overline{x}\ \phi(\overline{x}))^{\mathfrak{f}}$ to be $\forall\overline{x}\ (\bigwedge_i \delta_{\mathfrak{f}}(x_i) \to \phi(\overline{x})^{\mathfrak{f}})$. A *relative interpretation* $\mathfrak{f}\colon T \to U$ is a relative translation satisfying $T \vdash \phi \Rightarrow U \vdash \phi^{\mathfrak{f}}$ for all statements ϕ in T and $U \vdash \exists x\ \delta_{\mathfrak{f}}(x)$. Following Visser we define the following. For theories U, V, W in INT,

- the interpretation $\mathrm{id}_U\colon U \to U$ leaves relations unchanged;
- if $\mathfrak{f}\colon U \to V$ and $\mathfrak{g}\colon V \to W$ then $\mathfrak{f}\mathfrak{g}$ is defined by setting $R(\overline{x})^{\mathfrak{f}\mathfrak{g}}$ to be $(R(\overline{x})^{\mathfrak{f}})^{\mathfrak{g}}$.

Further, two interpretations $\mathfrak{f}, \mathfrak{g}\colon U \to V$ are considered equivalent if

[1] In the case that I is infinite, more care is required: in the case of Amphi-ZF any passage from one membership to the other is handled by the union axiom and schemes of replacement and separation. If I is infinite, however, this can be achieved in potentially different ways. One could include (many) additional replacement schemes, allowing the transfer of statements between different memberships. Alternatively having I as a separate sort within the model, and thus allowing quantification over I, may be preferred.

- $V \vdash \forall x\big(\delta_{\mathfrak{f}}(x) \leftrightarrow \delta_{\mathfrak{g}}(x)\big)$, and
- $V \vdash \forall \overline{x}\,\big(\bigwedge_i \delta_{\mathfrak{f}}(x_i) \to (\phi^{\mathfrak{f}}(\overline{x}) \leftrightarrow \phi^{\mathfrak{g}}(\overline{x}))\big)$ for all formulas ϕ in the language of U.

The morphisms in INT are these interpretations, modulo this equivalence (though generally we refer to specific interpretations). The interpretations $\mathfrak{f}\colon U \to V$ and $\mathfrak{g}\colon V \to U$ are said to be inverse to each other if their morphisms are inverse, i.e. if $\mathfrak{fg}$ and $\mathfrak{gf}$ are equivalent to 1_U and 1_V respectively.

4.2 Amphi-ZF

We work within the language $\mathscr{L}_{\in_{\mathrm{L}},\in_{\mathrm{R}}}$, having non-logical binary relation symbols $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$. Suppose $\mathscr{L}_2$ denotes the language with non-logical symbols $\in_{\mathrm{L}}, \in_{\mathrm{R}}$, both binary relations. We will not give much discussion of equality as defined by Conway, so throughout we use '=' to denote logical identity (so, for example, $\{0, 1|\} \neq \{1|\}$). For clarity and ease of expression we make the following definitions. Let x, y be games. Then

- $x \subseteq_{\mathrm{L}} y \Leftrightarrow \forall z \in_{\mathrm{L}} x\ (z \in_{\mathrm{L}} y)$;
- $x \subseteq_{\mathrm{R}} y \Leftrightarrow \forall z \in_{\mathrm{R}} x\ (z \in_{\mathrm{R}} y)$;
- $x \subseteq y \Leftrightarrow \forall z \bigwedge_{\mathrm{P}}(z \in_{\mathrm{P}} x \to z \in_{\mathrm{P}} y)$;
- $x \in^{\mathrm{L}}_{\mathrm{R}} y \Leftrightarrow (x \in_{\mathrm{L}} y \vee x \in_{\mathrm{R}} y)$.

Due to the symmetric nature of this system of axioms it is useful to adopt the following notation. We will write a sub- or superscript P (for example $\in_{\mathrm{P}}$) to indicate that one of L (Left) or R (Right) (eg. $\in_{\mathrm{L}}$ or $\in_{\mathrm{R}}$) is intended; however, in any expression P will only represent a particular side at any one time. For example, the string $\forall x \not\subseteq_{\mathrm{P}} 0\ \exists y(y \in_{\mathrm{P}} x)$ refers to either $\forall x \not\subseteq_{\mathrm{L}} 0\ \exists y(y \in_{\mathrm{L}} x)$ or $\forall x \not\subseteq_{\mathrm{R}} 0\ \exists y(y \in_{\mathrm{R}} x)$. Further, if $\phi_{\mathrm{L}}, \phi_{\mathrm{R}}$ are first-order formulas we define $\bigwedge_{\mathrm{P}} \phi_{\mathrm{P}}$ to be $\phi_{\mathrm{L}} \wedge \phi_{\mathrm{R}}$, and so on. In more complex statements it may be necessary to use additional letters Q, ... By $\not{\mathrm{P}}$ we denote the opponent of P. Now let us turn our attention to the axioms.

A0 (Zero Game Axiom)**.** There exists a zero game, i.e.

$$\exists x\, \forall y\big(y \notin_{\mathrm{L}} x \wedge y \notin_{\mathrm{R}} x\big).$$

A1 (Axiom of Extensionality)**.** Two games are equal if and only if their respective options are equal;

$$\forall x\, \forall y\Big(\bigwedge_{\mathrm{P}} \big(\forall z(z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)\big) \to x = y\Big).$$

Extensionality justifies our use of the familiar notation, for example using $\{u, v|x, y\}$ to denote the game with Left and Right options u, v and x, y respectively. Moreover if x is a game for which $y \in_{\mathrm{P}} x \Leftrightarrow \phi_{\mathrm{P}}(y)$, we may write $x = \{y\colon \phi_{\mathrm{L}}(y) \mid y\colon \phi_{\mathrm{R}}(y)\}$, or even $x = \{y\colon \phi_{\mathrm{P}}\}_{\mathrm{P}}$.

A2 (Pair-game Axiom)**.** If x, y are games there is a game with each as a left option;

$$\forall x\,\forall y\,\exists z(x \in_{\mathrm{L}} z \wedge y \in_{\mathrm{L}} z).$$

A3 (Replacement)**.** If $\phi(x, y, I)$ is a first-order formula (possibly with parameters), then

$$\forall I\left(\forall x \in^{\mathrm{L}}_{\mathrm{R}} I\ \exists! y\, \phi(x, y, I) \rightarrow \exists A\,\forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} A \leftrightarrow \exists x \in^{\mathrm{L}}_{\mathrm{R}} I\, \phi(x, z, I)\big)\right).$$

Notice the use of "$x \in^{\mathrm{L}}_{\mathrm{R}} I$" rather than "$x \in_{\mathrm{P}} I$"; this makes the set A more inclusive than might be expected. Therefore we allow side-specific restriction of elements in our separation scheme.

A4 (Separation)**.** For all first-order formulas $\phi_{\mathrm{L}}(u, v), \phi_{\mathrm{R}}(u, v)$ (possibly with parameters),

$$\forall x\,\exists y\,\forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} y \leftrightarrow z \in_{\mathrm{P}} x \wedge \phi_{\mathrm{P}}(x, z)\big).$$

We still require an axiom of union. There are several different types of union available, and so for clarity we designate a symbol for each:

$$\begin{aligned}
\biguplus x &= \{z\colon\ \exists y \in^{\mathrm{L}}_{\mathrm{R}} x(z \in^{\mathrm{L}}_{\mathrm{R}} y) | z\colon\ \exists y \in^{\mathrm{L}}_{\mathrm{R}} x(z \in^{\mathrm{L}}_{\mathrm{R}} y)\};\\
\bigcup x &= \{z\colon\ \exists y \in_{\mathrm{L}} x(z \in^{\mathrm{L}}_{\mathrm{R}} y) | z\colon\ \exists y \in_{\mathrm{R}} x(z \in^{\mathrm{L}}_{\mathrm{R}} y)\};\\
\boxplus x &= \{z\colon\ \exists y \in^{\mathrm{L}}_{\mathrm{R}} x(z \in_{\mathrm{L}} y) | z\colon\ \exists y \in^{\mathrm{L}}_{\mathrm{R}} x(z \in_{\mathrm{R}} y)\};\\
\bigsqcup x &= \{z\colon\ \exists y \in_{\mathrm{L}} x(z \in_{\mathrm{L}} y) | z\colon\ \exists y \in_{\mathrm{R}} x(z \in_{\mathrm{R}} y)\}.
\end{aligned}$$

Our Union axiom simply states that the largest of these, $\biguplus x$, exists for all games x. From this the other types can be constructed using separation.

A5 (Axiom of Union)**.**

$$\forall x\,\exists y\,\forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} y \leftrightarrow \exists w \in^{\mathrm{L}}_{\mathrm{R}} x(z \in^{\mathrm{L}}_{\mathrm{R}} w)\big).$$

We can also define unions and intersections of games along these lines. If x, y are games then $\{x, y|\}$ exists, and so $\{x^L, y^L|x^R, y^R\}$ does by Union and Separation. We call this game $x \cup y$. Analogously we define

$$x \cap y = \{z\colon\ z \in_{\mathrm{L}} x \wedge z \in_{\mathrm{L}} y | z\colon\ z \in_{\mathrm{R}} x \wedge z \in_{\mathrm{R}} y\},$$

and $x \setminus y = \{x^L\colon x^L \notin_{\mathrm{L}} y \mid x^R\colon x^R \notin_{\mathrm{R}} y\}$. Notice that we may also form successor games $s_{\mathrm{L}}(x) = \{x^L, x|x^R\} = x \cup \{x|\}$ and $s_{\mathrm{R}}(x) = x \cup \{|x\}$, and define $1 = s_{\mathrm{L}}(0)$, $2 = s_{\mathrm{L}}(1)$ and $-1 = s_{\mathrm{R}}(0)$, and so on. We define ordinals to be those amphisets which are $\in_{\mathrm{L}}$-transitive, well-ordered by $\in_{\mathrm{L}}$, and hereditarily right-empty. Our infinity axiom posits the existence of a set which is left- and right-inductive.

A6 (Infinity)**.**

$$\exists x \bigwedge_{\mathrm{P}} \big(0 \in_{\mathrm{P}} x \wedge \forall y \in_{\mathrm{P}} x(s_{\mathrm{P}}(y) \in_{\mathrm{P}} x)\big).$$

For a Foundation axiom there are several available choices, which vary in strength. We choose the following, which states precisely that $\in^{\text{L}}_{\text{R}}$ is a wellfounded relation;

A7 (Foundation)**.**

$$\forall x\big(\exists y(y \in^{\text{L}}_{\text{R}} x) \to \exists y \in^{\text{L}}_{\text{R}} x \,\forall z \in^{\text{L}}_{\text{R}} x(z \notin^{\text{L}}_{\text{R}} y)\big).$$

A8 (Power Game)**.**

$$\forall x \,\exists y \,\forall z\big(\bigwedge_{\text{P}} z \in_{\text{P}} y \leftrightarrow z \subseteq x\big).$$

By the Power Game and Separation axioms,

$$y = \{u \colon u \subseteq x \text{ and u is inductive }\}_{\text{P}}$$

is a game, as are $\{y^L|\}$ and $\{|y^R\}$. Defining the operator $\prod$ by

$$\prod u = \{w \colon \, \forall v \in_{\text{L}} u(w \in_{\text{L}} v)|w \colon \, \forall v \in_{\text{R}} u(w \in_{\text{R}} v)\},$$

we may define the game $\omega = \prod\{y^L|\}$. (Notice that, although $s_{\text{L}}(n) \leq n+1 \leq s_{\text{L}}(n)$ for all $n \in_{\text{L}} \omega$, $s_{\text{L}}(n)$ differs from Conway's definition of $n+1$, namely $\{n|\}$.) Using this we can define a transitive closure of a set, and proceed to prove an appropriate $\in^{\text{L}}_{\text{R}}$-induction principle, i.e.

$$\forall x\big(\forall y \in^{\text{L}}_{\text{R}} x \,\phi(y) \to \phi(x)\big) \to \forall x \phi(x)$$

for all formulas $\phi(u)$ (possibly with parameters). It may be useful to highlight here some related conventions that we have employed. An ordered pair (u, v) is simply the game $\{u \,|\, v\}$; a function is a game f with no right-members and only ordered pairs for Left options, subject to the condition that $\forall x \,\forall y \,\forall z\big((x, y) \in_{\text{L}} f \wedge (x, z) \in_{\text{L}} f \to y = z\big)$.

For each union type $\mathbb{U}$ of $\bigcup, \biguplus, \bigsqcup, \boxplus$ the transitive closure $\text{TC}(x, \mathbb{U})$ of a game x is defined recursively by setting $\text{TC}(x, 0, \mathbb{U}) = x$, and $\text{TC}(x, s_{\text{L}}(n), \mathbb{U}) = \mathbb{U}\,\text{TC}(x, n, (U)) \cup \text{TC}(x, n, \mathbb{U})$ for $n \in_{\text{L}} \omega$; then $\text{TC}(x, \mathbb{U})$ is the game

$$\{z \colon \, \exists n \in_{\text{L}} \omega \;\; \exists y \in_{\text{P}} \text{TC}(x, \mathbb{U}, n)(z \in_{\text{P}} y)\}_{\text{P}},$$

which exists by replacement. In particular $\text{TC}(x, \biguplus)$ is transitive in the relations $\in_{\text{LR}}, \in_{\text{L}}, \in_{\text{R}}, \in^{\text{L}}_{\text{R}}$, while $\text{TC}(x, \bigsqcup)$ is transitive in $\in_{\text{L}}, \in_{\text{R}}$, but not necessarily in $\in^{\text{L}}_{\text{R}}$.

By $\in^{\text{L}}_{\text{R}}$-recursion we can define the operations of negation, addition and multiplication. We can also specify what it means for a game to be less than, greater than or confused with another game.

4.3 Interpreting amphi-ZF in ZF

Working in ordinary ZF now, we use Quine's notion of ordered pairs [32] to define an interpretation of ZF_2. Significantly every set is a-Quine ordered pair, making it possible to interpret each set as an amphiset.

Definition 4.3.1. For all sets x we define

$$f_{\mathrm{L}}(x) = \{s(u)\colon u \in x \cap \omega\} \cup (x \setminus \omega),$$
$$f_{\mathrm{R}}(x) = \{0\} \cup \{s(u)\colon u \in x \cap \omega\} \cup (x \setminus \omega).$$

(Here $s(u)$ denotes the set-successor of u, $\{u\} \cup u$.) We can then define relations $\in_{\mathrm{L}}^{\mathfrak{v}}, \in_{\mathrm{R}}^{\mathfrak{v}}$ by $x \in_{\mathrm{P}}^{\mathfrak{v}} y \Leftrightarrow f_{\mathrm{P}}(x) \in y$. Formally, this means we define a translation $\mathfrak{v}\colon \mathrm{ZF}_2 \to \mathrm{ZF}$ by setting $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$ to be equivalent to $(f_{\mathrm{P}}(x) \in y)$; we will prove that $\mathfrak{v}$ is an interpretation.

It is useful to note that $f_{\mathrm{L}}, f_{\mathrm{R}}$ have a mutual left-inverse, defined by

$$g(x) = \{u \in \omega\colon s(u) \in x\} \cup (x \setminus \omega).$$

Before proving $\mathfrak{v}\colon \mathrm{ZF}_2 \to \mathrm{ZF}$ it will help to discuss the relations $(\in_{\mathrm{R}}^{\mathrm{L}})^{\mathfrak{v}}$ and $\subseteq^{\mathfrak{v}}$. Firstly, $x \sqsubseteq y$ if and only if $\forall z \bigwedge_{\mathrm{P}} (z \in_{\mathrm{P}} x \to z \in_{\mathrm{P}} y)$; so $x \sqsubseteq^{\mathfrak{v}} y$ if and only if $\forall z (\bigwedge_{\mathrm{P}} f_{\mathrm{P}}(z) \in x \to f_{\mathrm{P}}(z) \in y)$. Since the domains of $f_{\mathrm{L}}, f_{\mathrm{R}}$ together contain the entire set universe, this is equivalent to $x \subseteq y$.

For a function h, denote by $h[x]$ the set $\{h(y)\colon y \in x\}$. Then $(x \in_{\mathrm{R}}^{\mathrm{L}} y)^{\mathfrak{v}}$ is equivalent to the statement $f_{\mathrm{L}}(x) \in y \vee f_{\mathrm{R}}(x) \in y$; that is, $x \in g[y]$. So for all y, $g[y]$ is the set of sets x satisfying $(x \in_{\mathrm{R}}^{\mathrm{L}} y)^{\mathfrak{v}}$. Hence if ϕ is some $\mathscr{L}_{\in_{\mathrm{L}},\in_{\mathrm{R}}}$-formula, proving the existence of $\{y\colon \phi | y\colon \phi\}^{\mathfrak{v}}$ amounts to proving that of $g[\{y\colon \phi^{\mathfrak{v}}\}]$, which is true when $\{y\colon \phi^{\mathfrak{v}}\}$ exists, by replacement.

For instance, $z \in_{\mathrm{P}} \bigcup x$ is true if and only if $z \in_{\mathrm{R}}^{\mathrm{L}} y$ for some $y \in_{\mathrm{R}}^{\mathrm{L}} x$; thus for some y, $f_{\mathrm{P}}(z) \in y \in g[x]$, i.e. $f_{\mathrm{P}}(z) \in \bigcup g[x]$, and so $z \in g\big[\bigcup g[x]\big]$. The argument reverses, and so $\bigcup$ is interpreted by the function $g\big[\bigcup g[\cdot]\big]$.

These observations apply as well to the Replacement scheme. Let $\mathrm{Rep}(\phi)$ denote the amphi-replacement sentence

$$\forall I \left(\forall x \in_{\mathrm{R}}^{\mathrm{L}} I \; \exists! y \, \phi(x, y, I) \to \exists A \, \forall y \bigwedge_{\mathrm{P}} \big(y \in_{\mathrm{P}} A \leftrightarrow \exists x \in_{\mathrm{R}}^{\mathrm{L}} I \, \phi(x, y, I) \big) \right).$$

Assume $(\forall x \in_{\mathrm{R}}^{\mathrm{L}} I \; \exists! y \, \phi(x, y, I))^{\mathfrak{v}}$; that is, $\forall x \in g[I] \; \exists! y \, \phi^{\mathfrak{v}}(x, y, I)$. The statement

$$\left(\exists A \; \forall y \bigwedge_{\mathrm{P}} (y \in_{\mathrm{P}} A \leftrightarrow \exists x \in_{\mathrm{R}}^{\mathrm{L}} I \, \phi(x, y, I) \right)^{\mathfrak{v}}$$

is equivalent to

$$\exists A \; \forall y \bigwedge_{\mathrm{P}} (f_{\mathrm{P}}(y) \in A \leftrightarrow \exists x \in g[I] \, \phi^{\mathfrak{v}}(x, y, I)).$$

By replacement in ZF, $B = \{y\colon \exists x \in g[I] \, \phi^{\mathfrak{v}}(x, y, I)\}$ is a set. Let $A = f_{\mathrm{L}}[B] \cup f_{\mathrm{R}}[B]$; then $y \in_{\mathrm{P}} A$ if and only if $y \in B$, i.e. $\phi^{\mathfrak{v}}(x, y, I)$ for some x such that $(x \in_{\mathrm{R}}^{\mathrm{L}} I)^{\mathfrak{v}}$.

We now prove that $\mathrm{ZF} \vDash \mathrm{Found}^{\mathfrak{v}}$; the remaining axioms of ZF_2 are left to the reader.

First define a cumulative hierarchy of games in ZF as follows.

$$\mathscr{G}_0 = 0;$$
$$\mathscr{G}_{\alpha+1} = \{f_{\mathrm{L}}(z), f_{\mathrm{R}}(z) \colon z \subseteq \mathscr{G}_\alpha\}; \text{ and}$$
$$\mathscr{G}_\lambda = \bigcup_{\delta<\lambda} \mathscr{G}_\lambda \text{ for limit ordinals } \lambda.$$

Notice that this is exactly the interpretation in ZF of a particular cumulative hierarchy of games, since for all sets x, z, $z \subseteq x \Leftrightarrow z \sqsubseteq^{\mathfrak{v}} x$. By showing that every set is a member of this hierarchy, we can deduce the translation of Amphi-foundation in ZF.

Proposition 4.3.2. The sets $(\mathscr{G}_\alpha)_\alpha$ satisfy the following.

- If $\alpha < \beta$ then $\mathscr{G}_\alpha \subseteq \mathscr{G}_\beta$.
- For each ordinal α, $\mathscr{G}_\alpha$ is $\in$-transitive.
- Every set is in some $\mathscr{G}_\alpha$.

Proof. Each claim is proved by induction. Fix $\beta \in \mathbf{On}$ and assume that whenever $\gamma < \alpha < \beta$ we have $\mathscr{G}_\gamma \subseteq \mathscr{G}_\alpha$. If $\beta = \alpha + 1$, say, and $x \in \mathscr{G}_\alpha$ then for some $\gamma < \alpha$ there is $z \subseteq \mathscr{G}_\gamma$ such that $x = f_{\mathrm{P}}(z)$. As $z \subseteq \mathscr{G}_\gamma$, $z \subseteq \mathscr{G}_\alpha$, and so $x \in \mathscr{G}_{\alpha+1}$; thus $\mathscr{G}_\alpha \subseteq \mathscr{G}_{\alpha+1}$. The claim is clear when β is a limit.

To see the second claim suppose $\mathscr{G}_\alpha$ is transitive (in $\in$) for all $\alpha < \beta$. If $\beta = \alpha + 1$ and $y \in x \in \mathscr{G}_{\alpha+1}$ then $x = f_{\mathrm{P}}(z)$ for some $z \subseteq \mathscr{G}_\alpha$, and $y \in f_{\mathrm{P}}(z)$. If $y \notin \omega$ then $y \in z \subseteq \mathscr{G}_\alpha$, so by monotonicity $y \in \mathscr{G}_\beta$. If instead $y \in \omega$ then $y = 0$ (in which case $0 = f_{\mathrm{L}}(0) \in \mathscr{G}_1 \subseteq \mathscr{G}_\beta$) or $y = s(u)$ for some $u \in z \cap \omega$. Assuming the second case, $u \in \mathscr{G}_\alpha$, and so $u \subseteq \mathscr{G}_\alpha$ by transitivity. Since the successor map and f_{R} coincide on ω, $y = f_{\mathrm{R}}(u) \in \mathscr{G}_\beta$. If instead β is a limit ordinal then $\mathscr{G}_\lambda$ is a union of transitive sets, and hence the claim.

For the final claim, suppose $x \subseteq \mathscr{G}_\alpha$, but $x \notin \mathscr{G}_{\alpha+1}$. Then $g(x) \not\subseteq \mathscr{G}_\alpha$. Let $y \in g(x) \setminus \mathscr{G}_\alpha$. If $y \notin \omega$ then $y \in x \subseteq \mathscr{G}_\alpha$, a contradiction. Therefore $y \in \omega$, so $y \in s(y) \in x \subseteq \mathscr{G}_\alpha$. By transitivity of $\mathscr{G}_\alpha$, $y \in \mathscr{G}_\alpha$ – a contradiction. Therefore $x \subseteq \mathscr{G}_\alpha \Rightarrow x \in \mathscr{G}_{\alpha+1}$, i.e. $\mathscr{P}(\mathscr{G}_\alpha) \subseteq \mathscr{G}_{\alpha+1}$ for all α. In particular as $V_0 = \mathscr{G}_0$ we have that $V_\alpha \subseteq \mathscr{G}_\alpha$ for all α, where the sets V_α form the usual cumulative hierarchy. The claim follows. □

Corollary 4.3.3. $\mathrm{ZF} \vdash \mathrm{Found}^{\mathfrak{v}}$.

Proof. Define the "game rank" $\mathrm{gr}(y)$ of a set y to be the least ordinal α such that $y \subseteq \mathscr{G}_\alpha$. Let x be an arbitrary set, and pick $y \in^{\mathrm{L}}_{\mathrm{R}} x$ of minimal game rank α. Supposing there exists z such that $z \in^{\mathrm{L}}_{\mathrm{R}} x \wedge z \in^{\mathrm{L}}_{\mathrm{R}} y$, some $f_{\mathrm{P}}(z) \in y$ and so $z \subseteq \mathscr{G}_\beta$ for some $\beta < \alpha$. Hence $\mathrm{gr}(z) \le \beta < \mathrm{gr}(y)$, contradicting our choice of y. □

The remaining axioms are easily verified, giving the following.

Theorem 4.3.4. $\mathfrak{v} \colon \mathrm{ZF}_2 \to \mathrm{ZF}$.

4.4 Interpreting ZF in amphi-ZF

Firstly we observe that ZF can be interpreted easily in a subclass of any model $\mathscr{G}$ of amphi-ZF. This enables us to define a membership $\in^{\mathfrak{g}}$ in $\mathscr{G}$, essentially reflecting the behaviour of $\in^{\mathfrak{g}}$ on this subclass. Define $\mathscr{G}_{\mathrm{L}}$ to be the subclass of games which are hereditarily right-empty; formally we let $\delta_{\mathfrak{l}}(x)$ read $\mathrm{TC}(x, ⨃) \subseteq_{\mathrm{R}} 0$, for an interpretation $\mathfrak{l}\colon \mathrm{ZF} \to \mathrm{ZF}_2$ given by $(x \in y)^{\mathfrak{l}} = (x \in_{\mathrm{L}} y)$.

Proposition 4.4.1. The substructure $(\mathscr{G}_L, \in_{\mathrm{L}})$ satisfies the axioms of ZF; therefore $\mathfrak{l}\colon \mathrm{ZF} \to \mathrm{ZF}_2$.

In order to find an interpretation $\mathfrak{g}\colon \mathrm{ZF} \to \mathrm{ZF}_2$ whose domain contains all games we construct a (definable) bijection $F\colon \mathscr{G}_L \to \mathscr{G}$, and mirror the behaviour of $\in$ in $\mathscr{G}$. This bijection can be defined in such a way that the new interpretation $\mathfrak{g}$ is inverse to $\mathfrak{v}$.

The functions $f^{\mathfrak{l}}_{\mathrm{L}}, f^{\mathfrak{l}}_{\mathrm{R}}, g^{\mathfrak{l}}$ are determined uniquely and given by

$$\begin{aligned}
f^{\mathfrak{l}}_{\mathrm{L}}(x) &= \{s_{\mathrm{L}}(u)\colon u \in_{\mathrm{L}} x \cap \omega |\} \cup (x \setminus \omega);\\
f^{\mathfrak{l}}_{\mathrm{R}}(x) &= \{0\,|\,\} \cup f^{\mathfrak{l}}_{\mathrm{L}}(x);\\
g^{\mathfrak{l}}(x) &= \{u\colon s_{\mathrm{L}}(u) \in_{\mathrm{L}} x \cap \omega\,|\,\} \cap (x \setminus \omega).
\end{aligned}$$

(throughout this section we use ω to denote the game $\{0, 1, \ldots |\}$ in $\mathscr{G}_L$). The appropriate definition of F is then rather straightforward: if x is a set then above we see it as the game $\{y\colon f_{\mathrm{L}}(y) \in x | y\colon f_{\mathrm{R}}(y) \in x\}$, where each set y is already camouflaged as a game. Thus we define

$$F(x) = \{F(y)\colon f^{\mathfrak{l}}_{\mathrm{L}}(y) \in^{\mathfrak{l}} x | F(y)\colon f^{\mathfrak{l}}_{\mathrm{R}}(y) \in^{\mathfrak{l}} x\}.$$

The inverse is easily described: we define, for all games x,

$$G(x) = \{f^{\mathfrak{l}}_{\mathrm{P}}(G(y))\colon y \in_{\mathrm{P}} x|\}.$$

Proving that $G = F^{-1}$, however, requires some work. As will be the case throughout this chapter, proofs are simplified by first proving a result restricted to the natural numbers[1].

Lemma 4.4.2. For all $n \in_{\mathrm{L}} \omega$, $G \circ F(n) = F \circ G(n) = n$.

Proof. Notice that $f^{\mathfrak{l}}_{\mathrm{L}}(y) \in_{\mathrm{L}} n$ if and only if $n > 0$ and $y = 0$. Further, $f^{\mathfrak{l}}_{\mathrm{R}} \restriction \omega = s_{\mathrm{L}} \restriction \omega$ so $f^{\mathfrak{l}}_{\mathrm{R}}(y) \in_{\mathrm{L}} n$ if and only if $n > 1$ and $y \leq n - 2$. Therefore,

$$F(n) = \{F(0)|F(0), \ldots, F(n-2)\}.$$

It is easily seen that $F(0) = G(0) = 0$ and $F(1) = G(1) = 1$. Assume, therefore, that

[1] This might be expected, given our dependence on the functions f_{P}. Essentially a normal inductive hypothesis must be strengthened so that some property holds, not just for all members of a set x, but for all members of $g[x]$ as well. We are able to circumnavigate this overly technical exercise by first proving the same statements for all natural numbers (as realised in the theory). As a consequence of their definitions, each of the functions $F, G, f_{\mathrm{L}}, f_{\mathrm{R}}$ (interpreted in each language) is a homomorphism with respect to the appropriate operations of union, intersection and difference.

$n \geq 2$ and for all $m < n$, $G \circ F(m) = F \circ G(m) = m$. Then

$$\begin{aligned} G \circ F(n) &= G(\{F(0) \mid F(0), \ldots, F(n-2)\}) \\ &= \{f^{\mathfrak{l}}_{\mathrm{L}}(0), f^{\mathfrak{l}}_{\mathrm{R}}(0), \ldots, f^{\mathfrak{l}}_{\mathrm{R}}(n-2) \mid \} \\ &= n, \end{aligned}$$

by induction. In the other direction,

$$\begin{aligned} G(n) &= \{f^{\mathfrak{l}}_{\mathrm{P}}(G(y)) \colon y \in_{\mathrm{P}} n \mid \} \\ &= \{f^{\mathfrak{l}}_{\mathrm{L}}(G(m)) \colon m < n \mid \}, \end{aligned}$$

so that

$$\begin{aligned} F \circ G(n) &= \{F(y) \colon f^{\mathfrak{l}}_{\mathrm{L}}(y) \in_{\mathrm{L}} G(n) \mid \} \\ &= \{F \circ G(m) \colon m < n \mid \} \\ &= n, \end{aligned}$$

by induction. □

Proposition 4.4.3. For all x, $F(G(x)) = G(F(x)) = x$.

Proof. Suppose $\forall x \in^{\mathrm{L}}_{\mathrm{R}} \mathrm{TC}(y, \biguplus)\, (F(G(x)) = G(F(x)) = x)$. Then

$$\begin{aligned} F(G(y)) &= \{F(w) \colon f^{\mathfrak{l}}_{\mathrm{P}}(w) \in_{\mathrm{L}} G(y)\}_{\mathrm{P}} \\ &= \{F(G(z)) \colon z \in_{\mathrm{P}} y\}_{\mathrm{P}} \\ &= y \end{aligned}$$

by induction. If $y \in_{\mathrm{L}} \mathscr{G}_{\mathrm{L}}$, then

$$\begin{aligned} G(F(y)) &= \{f^{\mathfrak{l}}_{\mathrm{P}}(G(z)) \colon z \in_{\mathrm{P}} F(y) \mid \} \\ &= \{f^{\mathfrak{l}}_{\mathrm{P}}(G(z)) \colon z = F(w) \text{ for some } w \text{ such that } f^{\mathfrak{l}}_{\mathrm{P}}(w) \in_{\mathrm{L}} y \mid \} \end{aligned}$$

We cannot use the inductive hypothesis, since such a w may not be a member of y. However, we know that

$$\begin{aligned} G \circ F(y \cap \omega) &= \{f^{\mathfrak{l}}_{\mathrm{P}} \circ G \circ F(n) \colon f^{\mathfrak{l}}_{\mathrm{P}}(n) \in_{\mathrm{L}} y \cap \omega \mid \} \\ &= \{f^{\mathfrak{l}}_{\mathrm{P}}(n) \colon f^{\mathfrak{l}}_{\mathrm{P}}(n) \in_{\mathrm{L}} y \cap \omega \mid \} \\ &= y \cap \omega, \end{aligned}$$

and by the inductive hypothesis, $G \circ F(y \setminus \omega) = y \setminus \omega$ in the same way. Therefore $G \circ F(y) = y$. □

Definition 4.4.4. Let x, y be amphisets. We write $x \in^{\mathfrak{g}} y$ if and only if $G(x) \in_{\mathrm{L}} G(y)$. By $\mathfrak{g}$ we denote the corresponding interpretation.

Notice that by definition $x \in^{\mathfrak{g}} y \leftrightarrow G(x) \in^{\mathfrak{l}} G(y)$. It follows that

$$\mathrm{ZF}_2 \vdash \forall x_0, \ldots, x_n\, (\phi^{\mathfrak{g}}(x_0, \ldots, x_n) \leftrightarrow \phi^{\mathfrak{l}}(G(x_0), \ldots, G(x_n)).$$

An immediate consequence is the following.

Theorem 4.4.5. $\mathfrak{g}\colon \mathrm{ZF} \to \mathrm{ZF}_2$.

In Visser's terms [34, p.11] the formula $\phi(x, y)$ reading $F(x) = y$ induces an i-map $\phi\colon \mathfrak{l} \to \mathfrak{g}$.

4.5 Showing $\mathfrak{g} = \mathfrak{v}^{-1}$

4.5.1 Showing $\mathfrak{v}\mathfrak{g} = 1_{\mathrm{AZF}}$

Before proving that $\mathfrak{v}$ and $\mathfrak{g}$ are mutual inverses, we require two technical lemmas. To reduce confusion we shall also introduce some new notation. There are many concepts common to both theories, most significantly different natural numbers, as well as the ordinal ω. Henceforth, we use a subscript AZF or ZF to indicate which theory is realising the object. For instance, for the natural number 0 (realised as the empty container in each theory) we let 0_{AZF} be the game $\{\,|\,\}$, and 0_{ZF} the set $\{\,\}$. Similarly we may write $A^{\mathfrak{g}}_{\mathrm{ZF}}$ for axioms of ZF interpreted in $\mathscr{L}_{\in_{\mathrm{L}},\in_{\mathrm{R}}}$ via $\mathfrak{g}$. Although somewhat cumbersome, this notation does much to clarify the following arguments.

Lemma 4.5.1. For each natural number n,

- $F(s_{\mathrm{L}}(n_{\mathrm{AZF}})) = s^{\mathfrak{g}}_{\mathrm{ZF}}(F(n_{\mathrm{AZF}}))$;
- $F(n_{\mathrm{AZF}}) = n^{\mathfrak{g}}_{\mathrm{ZF}}$.

Proof. We use $\mathrm{Ext}^{\mathfrak{g}}_{\mathrm{ZF}}$ to prove the first point. If $x \in^{\mathfrak{g}} F(s_{\mathrm{L}}(u))$ then $G(x) \in_{\mathrm{L}} s_{\mathrm{L}}(u)$, so $G(x) = u$ or $G(x) \in_{\mathrm{L}} u$. Thus $x = F(u) \vee x \in^{\mathfrak{g}} F(u)$, so $x \in^{\mathfrak{g}} s^{\mathfrak{g}}_{\mathrm{ZF}}(F(u))$ by definition. The argument reverses, whence $F(s_{\mathrm{L}}(u)) = s^{\mathfrak{g}}_{\mathrm{ZF}}(F(u))$ for all $u \in_{\mathrm{L}} \omega$.

Notice that $F(0_{\mathrm{AZF}}) = 0_{\mathrm{AZF}} = 0^{\mathfrak{g}}_{\mathrm{ZF}}$; the previous result then gives us an inductive proof that $F(n_{\mathrm{AZF}}) = n^{\mathfrak{g}}_{\mathrm{ZF}}$ for all natural numbers n. $\square$

Lemma 4.5.2. It is provable within ZF_2 that $G(f^{\mathfrak{g}}_{\mathrm{P}}(x)) = f^{\mathfrak{l}}_{\mathrm{P}}(G(x))$ for all x. Consequently, for all x, $G(x) = \{G(f^{\mathfrak{g}}_{\mathrm{P}}(y))\colon y \in_{\mathrm{P}} x\,|\,\}$.

Proof. First we note that on each side the game is right-empty, since G and $f^{\mathfrak{l}}_{\mathrm{P}}$ are both functions mapping on to the class of such games. We deal with three distinct cases.

Firstly, suppose $0_{\mathrm{AZF}} \in_{\mathrm{L}} G(f^{\mathfrak{g}}_{\mathrm{P}}(x))$; then $0_{\mathrm{AZF}} = f^{\mathfrak{l}}_{\mathrm{Q}}(w)$ for some w with $F(w) \in_{\mathrm{Q}} f^{\mathfrak{g}}_{\mathrm{Q}}(x)$. This forces $\mathrm{Q} = \mathrm{L}$ and $w = 0_{\mathrm{AZF}}$, so $0_{\mathrm{AZF}} \in_{\mathrm{L}} f^{\mathfrak{g}}_{\mathrm{P}}(x)$. Therefore $\mathrm{P} = \mathrm{R}$. Since $0_{\mathrm{AZF}} \in_{\mathrm{L}} f^{\mathfrak{l}}_{\mathrm{R}}(t)$ for all games t, we also have $0_{\mathrm{AZF}} \in_{\mathrm{L}} f^{\mathfrak{l}}_{\mathrm{R}}(G(x))$.

Conversely if $0_{\mathrm{AZF}} \in_{\mathrm{L}} f^{\mathfrak{l}}_{\mathrm{P}}(G(x))$, $\mathrm{P} = \mathrm{R}$ again. Since $0_{\mathrm{AZF}} = F(0_{\mathrm{AZF}}) \in^{\mathfrak{g}} f^{\mathfrak{g}}_{\mathrm{R}}(x)$, $0_{\mathrm{AZF}} \in_{\mathrm{L}} G(f^{\mathfrak{g}}_{\mathrm{P}}(x))$. Hence $0_{\mathrm{AZF}} \in_{\mathrm{L}} f^{\mathfrak{l}}_{\mathrm{P}}(G(x))$ if and only if $0_{\mathrm{AZF}} \in_{\mathrm{L}} G(f^{\mathfrak{g}}_{\mathrm{P}}(x))$.

For the second case, suppose $u \notin_{\mathrm{L}} \omega_{\mathrm{AZF}}$. If $u \in_{\mathrm{L}} G(f^{\mathfrak{g}}_{\mathrm{P}}(x))$ then $F(u) \in^{\mathfrak{g}} f^{\mathfrak{g}}_{\mathrm{P}}(x)$. Since $u \notin_{\mathrm{L}} \omega_{\mathrm{AZF}}$, $F(u) \notin^{\mathfrak{g}} \omega^{\mathfrak{g}}_{\mathrm{ZF}}$, and so $F(u) \in^{\mathfrak{g}} x$; therefore $u \in_{\mathrm{L}} G(x)$. As $u \notin_{\mathrm{L}} \omega_{\mathrm{AZF}}$, $u \in_{\mathrm{L}} f^{\mathfrak{l}}_{\mathrm{P}}(G(x))$. Conversely if $u \in_{\mathrm{L}} f^{\mathfrak{l}}_{\mathrm{P}}(G(x))$, $u \in_{\mathrm{L}} G(x)$ so $F(u) \in^{\mathfrak{g}} x$. Since $F(u) \notin^{\mathfrak{g}} \omega^{\mathfrak{g}}_{\mathrm{ZF}}$, $F(u) \in^{\mathfrak{g}} f^{\mathfrak{g}}_{\mathrm{P}}(x)$, hence $u \in_{\mathrm{L}} G(f^{\mathfrak{g}}_{\mathrm{P}}(x))$.

For the final case, assume u is a nonzero natural number in amphi-ZF. If $u \in_{\mathrm{L}} G(f_{\mathrm{P}}^{\mathfrak{g}}(x))$ then $F(u) \in^{\mathfrak{g}} f_{\mathrm{P}}^{\mathfrak{g}}(x)$. Since $0_{\mathrm{ZF}} \neq F(u) \in^{\mathfrak{g}} \omega_{\mathrm{ZF}}^{\mathfrak{g}}$, $F(u) = s_{\mathrm{ZF}}^{\mathfrak{g}}(m_{\mathrm{ZF}}^{\mathfrak{g}})$ for some natural number m such that $m_{\mathrm{ZF}}^{\mathfrak{g}} \in^{\mathfrak{g}} x$. By lemma 4.5.1 $m_{\mathrm{ZF}}^{\mathfrak{g}} = F(v)$ for some $v \in_{\mathrm{L}} \omega_{\mathrm{AZF}}$, and moreover $u = s_{\mathrm{L}}(v)$. As $m_{\mathrm{ZF}}^{\mathfrak{g}} \in^{\mathfrak{g}} x$, $v = G(m_{\mathrm{ZF}}^{\mathfrak{g}}) \in_{\mathrm{L}} G(x)$, and so $u = s_{\mathrm{L}}(v) \in_{\mathrm{L}} f_{\mathrm{P}}^{\mathfrak{l}}(G(x))$.

Conversely if $u \in_{\mathrm{L}} f_{\mathrm{P}}^{\mathfrak{l}}(G(x))$ then the predecessor v of u satisfies $v \in_{\mathrm{L}} G(x)$, hence $F(v) \in^{\mathfrak{g}} x$. Since $v \in_{\mathrm{L}} \omega_{\mathrm{AZF}}$, $F(v) \in^{\mathfrak{g}} \omega_{\mathrm{ZF}}^{\mathfrak{g}}$, whence $F(u) = s_{\mathrm{ZF}}^{\mathfrak{g}}(F(v)) \in^{\mathfrak{g}} f_{\mathrm{P}}^{\mathfrak{g}}(x)$; therefore $u = G(s_{\mathrm{ZF}}^{\mathfrak{g}}(F(v))) \in_{\mathrm{L}} G(f_{\mathrm{P}}^{\mathfrak{g}}(x))$. $\square$

Theorem 4.5.3. In amphi-ZF it is provable that $\forall x, y \bigwedge_{\mathrm{P}} (x \in_{\mathrm{P}} y \leftrightarrow x \in_{\mathrm{P}}^{\mathfrak{g}} y)$. Therefore $\mathfrak{vg} = 1_{\mathrm{AZF}}$ in INT.

Proof. Indeed,

$$\begin{aligned} x \in_{\mathrm{P}}^{\mathfrak{vg}} y &\leftrightarrow (f_{\mathrm{P}}(x) \in y)^{\mathfrak{g}} \\ &\leftrightarrow G(f_{\mathrm{P}}^{\mathfrak{g}}(x)) \in_{\mathrm{L}} G(y) \\ &\leftrightarrow x \in_{\mathrm{P}} y, \end{aligned}$$

by lemma 4.5.2. $\square$

4.5.2 Proving $\mathfrak{gv} = 1_{\mathrm{ZF}}$

Remark 4.5.4. Before we proceed it will be helpful to clarify the behaviour of the relation $\in^{\mathfrak{gv}}$. Lemma 4.5.2 implies

$$G^{\mathfrak{v}}(x) = \{f_{\mathrm{L}} \circ G^{\mathfrak{v}} \circ f_{\mathrm{P}}^{\mathfrak{gv}}(y)\colon f_{\mathrm{P}}(y) \in x\},$$

and so we deduce

$$\begin{aligned} u \in^{\mathfrak{g}v} x &\leftrightarrow (u \in^{\mathfrak{g}} x)^{\mathfrak{v}} \\ &\leftrightarrow (G(u) \in_{\mathrm{L}} G(x))^{\mathfrak{v}} \\ &\leftrightarrow f_{\mathrm{L}}(G^{\mathfrak{v}}(u)) \in G^{\mathfrak{v}}(x) \\ &\leftrightarrow \exists y \, (f_{\mathrm{P}}(y) \in x \wedge u = f_{\mathrm{P}}^{\mathfrak{gv}}(y). \end{aligned}$$

We now show that $\in^{\mathfrak{gv}}$ behaves just like $\in$ regarding the natural numbers in ZF.

Lemma 4.5.5. For all natural numbers n,

1. $n_{\mathrm{ZF}}^{\mathfrak{gv}} = n_{\mathrm{ZF}}$;
2. $\forall x(x \in^{\mathfrak{gv}} n_{\mathrm{ZF}} \leftrightarrow x \in n_{\mathrm{ZF}})$;
3. $\forall x(n_{\mathrm{ZF}} \in^{\mathfrak{gv}} x \leftrightarrow n_{\mathrm{ZF}} \in x)$.

It follows that the same points hold, with n replaced by ω.

Proof. Notice that, working in ZF_2, $0^{\mathfrak{g}}_{\mathrm{ZF}} = 0_{\mathrm{AZF}}$, $1^{\mathfrak{g}}_{\mathrm{ZF}} = 1_{\mathrm{AZF}}$, and

$$\begin{aligned} n^{\mathfrak{g}}_{\mathrm{ZF}} &= F(n_{\mathrm{AZF}}) \\ &= \{F(y)\colon f^{\mathfrak{l}}_{\mathrm{L}}(y) \in_{\mathrm{L}} n_{\mathrm{AZF}} \mid F(y)\colon f^{\mathfrak{l}}_{\mathrm{R}}(y) \in_{\mathrm{L}} n_{\mathrm{AZF}}\} \\ &= \{F(0_{\mathrm{AZF}}) \mid F(0_{\mathrm{AZF}}), \ldots, F((n-2)_{\mathrm{AZF}})\} \\ &= \{0^{\mathfrak{g}}_{\mathrm{ZF}} \mid 0^{\mathfrak{g}}_{\mathrm{ZF}}, \ldots, (n-2)^{\mathfrak{g}}_{\mathrm{ZF}}\} \end{aligned}$$

for $n > 1$, by induction. Therefore, back in ZF, $0^{\mathfrak{gv}}_{\mathrm{ZF}} = 0_{\mathrm{ZF}}$, $1^{\mathfrak{gv}}_{\mathrm{ZF}} = 1_{\mathrm{ZF}}$, and

$$\begin{aligned} n^{\mathfrak{gv}}_{\mathrm{ZF}} &= \{f_{\mathrm{L}}(0_{\mathrm{ZF}})\} \cup \{f_{\mathrm{R}}(0_{\mathrm{ZF}}, \ldots, f_{\mathrm{R}}((n-1)_{\mathrm{ZF}})\} && (4.1) \\ &= n_{\mathrm{ZF}}, && (4.2) \end{aligned}$$

by induction and since $f_{\mathrm{R}} \restriction \omega_{\mathrm{ZF}}$ is the successor function.

The second point is easily proved by induction: suppose it holds for all $k \leq n$; then

$$\begin{aligned} x \in^{\mathfrak{gv}} (n+1)^{\mathfrak{gv}}_{\mathrm{ZF}} &\Leftrightarrow x \in^{\mathfrak{gv}} n^{\mathfrak{gv}}_{\mathrm{ZF}} \vee x = n^{\mathfrak{gv}}_{\mathrm{ZF}} && \text{(by definition of successor)} \\ &\Leftrightarrow x \in^{\mathfrak{gv}} n_{\mathrm{ZF}} \vee x = n_{\mathrm{ZF}} && \text{(by part 1)} \\ &\Leftrightarrow x \in n_{\mathrm{ZF}} \vee x = n_{\mathrm{ZF}} && \text{(by induction)} \\ &\Leftrightarrow x \in (n+1)_{\mathrm{ZF}} && \text{(by definition of successor).} \end{aligned}$$

For the final statement, it is easily seen that $f^{\mathfrak{gv}}_{\mathrm{L}}(0_{\mathrm{ZF}}) = 0_{\mathrm{ZF}}$ and $f^{\mathfrak{gv}}_{\mathrm{R}}(0_{\mathrm{ZF}}) = \{0_{\mathrm{ZF}}\}$. Take an arbitrary set x. If $0_{\mathrm{ZF}} \in^{\mathfrak{gv}} x$ then $0_{\mathrm{ZF}} = f^{\mathfrak{gv}}_{\mathrm{L}}(t)$ for some t satisfying $f_{\mathrm{P}}(t) \in x$, by remark 4.5.4. Since $f^{\mathfrak{gv}}_{\mathrm{L}}(0_{\mathrm{ZF}}) = 0_{\mathrm{ZF}}$, $t = 0_{\mathrm{ZF}}$ and $0_{\mathrm{ZF}} = f_{\mathrm{L}}(t) \in x$. Conversely if $0_{\mathrm{ZF}} \in x$, then $0_{\mathrm{ZF}} = f^{\mathfrak{gv}}_{\mathrm{L}}(0_{\mathrm{ZF}}) = f_{\mathrm{L}}(0_{\mathrm{ZF}}) \in x$, so that $0_{\mathrm{ZF}} \in^{\mathfrak{gv}} x$. This proves the statement for $n = 0_{\mathrm{ZF}}$.

If $n_{\mathrm{ZF}} \in^{\mathfrak{gv}} x$, $n_{\mathrm{ZF}} = f^{\mathfrak{gv}}_{\mathrm{P}}(t)$ for some t satisfying $f_{\mathrm{P}}(t) \in x$. Either $\mathrm{P} = \mathrm{R}$, in which case $n^{\mathfrak{gv}}_{\mathrm{ZF}} = f^{\mathfrak{gv}}_{\mathrm{R}}(t)$, whence $t = (n-1)_{\mathrm{ZF}}$ and $n_{\mathrm{ZF}} = f_{\mathrm{R}}(t) \in x$; or $\mathrm{P} = \mathrm{L}$, which forces $t = n_{\mathrm{ZF}} = 0_{\mathrm{ZF}} \in x$. If $n_{\mathrm{ZF}} \notin^{\mathfrak{gv}} x$, find t such that $f^{\mathfrak{gv}}_{\mathrm{P}}(t) = n_{\mathrm{ZF}}$. Then $n_{\mathrm{ZF}} = f_{\mathrm{P}}(t)$ by the same argument, and so $n_{\mathrm{ZF}} \notin x$. □

Corollary 4.5.6. For all natural numbers n, $f^{\mathfrak{gv}}_{\mathrm{P}}(n_{\mathrm{ZF}}) = f_{\mathrm{P}}(n_{\mathrm{ZF}})$. Further if $f_{\mathrm{P}}(u) \in \omega_{\mathrm{ZF}}$ or $f^{\mathfrak{gv}}_{\mathrm{P}}(u) \in \omega_{\mathrm{ZF}}$ then $f_{\mathrm{P}}(u) = f^{\mathfrak{gv}}_{\mathrm{P}}(u)$.

In light of Lemma 4.5.5 we may drop the subscripts denoting which theory an object is interpreted in. Finally we can prove that $\mathfrak{gv} = 1_{\mathrm{ZF}}$.

Theorem 4.5.7. Assume ZF. For all sets x, y, $x \in y \leftrightarrow x \in^{\mathfrak{gv}} y$. Therefore $\mathfrak{gv} = 1_{\mathrm{ZF}}$.

Proof. Define formulas

$$\begin{aligned} &\phi(x) \equiv \forall z\, (z \in x \leftrightarrow z \in^{\mathfrak{gv}} x); \\ &\psi(x) \equiv \forall z\, (x \in x \leftrightarrow x \in^{\mathfrak{gv}} z); \\ &\theta(x) \equiv \forall z \bigwedge_{\mathrm{P}} (x = f_{\mathrm{P}}(z) \leftrightarrow x = f^{\mathfrak{gv}}_{\mathrm{P}}(z))\sigma(x) \qquad \equiv \bigwedge_{\mathrm{P}} (f_{\mathrm{P}}(x) = f^{\mathfrak{gv}}_{\mathrm{P}}(x)). \end{aligned}$$

Assume $\forall x (\mathrm{rk}(x) < \mathrm{rk}(y) \rightarrow \phi(x) \wedge \psi(x) \wedge \theta(x) \wedge \sigma(x))$; we prove that y satisfies each formula.

First we prove $\phi(y)$. Fix any x. If $x \in \omega$ then by Lemma 4.5.5 $x \in y \leftrightarrow x \in^{\mathfrak{gv}} y$. Assume, therefore, that $x \notin \omega$. Suppose first that $x \in^{\mathfrak{gv}} y$. Then $x = f_{\mathrm{Q}}^{\mathfrak{gv}}(z)$ for some z such that $f_{\mathrm{Q}}(z) \in y$. Since $\mathrm{rk}(f_{\mathrm{Q}}(z)) < \mathrm{rk}(y)$, by $\theta(f_{\mathrm{Q}}(z))$, $x = f_{\mathrm{Q}}(z) \in y$. Suppose now that $x \in y$. Take z such that $x = f_{\mathrm{Q}}(z)$. Then by $\theta(x)$, $x = f_{\mathrm{Q}}^{\mathfrak{gv}}(z)$, hence $x \in^{\mathfrak{gv}} y$. This proves $\phi(y)$.

We now consider $\theta(y)$, with the following cases.

- Suppose $y = f_{\mathrm{P}}(z)$ for some z. If $u \in \omega$ it is easily shown that $u \in y \leftrightarrow u \in f_{\mathrm{P}}^{\mathfrak{gv}}(z)$. Assume $u \notin \omega$, and suppose first that $u \in y$. Then $u \in z$, so $\mathrm{rk}(u) < \mathrm{rk}(z) \leq \mathrm{rk}(y)$. By $\psi(u)$, $u \in^{\mathfrak{gv}} z$, whence $u \in^{\mathfrak{gv}} f_{\mathrm{P}}^{\mathfrak{gv}}(z)$. By $\psi(u)$ again, $u \in f_{\mathrm{P}}^{\mathfrak{gv}}(z)$.

  Conversely suppose $u \notin \omega$ and $u \in f_{\mathrm{P}}^{\mathfrak{gv}}(z)$. Take v such that $u = f_{\mathrm{Q}}(v)$; then $w = f_{\mathrm{Q}}^{\mathfrak{gv}}(v) \in^{\mathfrak{gv}} f_{\mathrm{P}}^{\mathfrak{gv}}(z)$. If $w \in \omega$, then by corollary 4.5.6 $w = u$ – contradicting $u \notin \omega$. So $w \notin \omega$, and $w \in^{\mathfrak{gv}} z$. Since $\mathrm{rk}(w) < \mathrm{rk}(z) \leq \mathrm{rk}(y)$, $\theta(w)$ implies $u = w$ and $\psi(w)$ implies $w \in z$. Thus $u = w \in f_{\mathrm{P}}(z) = y$.

- Suppose now that $y = f_{\mathrm{P}}^{\mathfrak{gv}}(z)$. Again the cases where $u \in \omega$ are easy, so we assume $u \notin \omega$. Suppose first that $u \in^{\mathfrak{gv}} y$. As $u \notin \omega$, $u \in^{\mathfrak{gv}} z$. By $\phi(y)$, $u \in y$, and so $\psi(u)$ is true. Thus $u \in z$. Since $u \notin \omega$, $u \in f_{\mathrm{P}}(z)$.

  Conversely suppose $u \in^{\mathfrak{gv}} f_{\mathrm{P}}(z)$. Then $u = f_{\mathrm{Q}}^{\mathfrak{gv}}(v)$ for some v such that $w = f_{\mathrm{Q}}(v) \in f_{\mathrm{P}}(z)$. If $w \in \omega$, $u = w$ by corollary 4.5.6, contradicting $u \notin \omega$. Therefore $w \notin \omega$, and $w \in z$. But then $u \in^{\mathfrak{gv}} z$, whence $u \in^{\mathfrak{gv}} y$. By $\mathrm{Ext}^{\mathfrak{gv}}$, $y = f_{\mathrm{P}}(z)$.

The remaining formulas, ψ and σ, are much simpler. To see $\psi(y)$, first suppose $y \in^{\mathfrak{gv}} x$. Then $y = f_{\mathrm{Q}}^{\mathfrak{gv}}(z)$ for some z with $f_{\mathrm{Q}}(z) \in x$. By $\theta(y)$, $y = f_{\mathrm{Q}}(z) \in x$. If instead we start with $y \in x$, find z such that $y = f_{\mathrm{Q}}(z)$. By $\theta(y)$, $y = f_{\mathrm{Q}}^{\mathfrak{gv}}(z) \in^{\mathfrak{gv}} x$.

Tos see $\sigma(y)$, first suppose that $u \in f_{\mathrm{P}}^{\mathfrak{gv}}(y)$. There is a set v such that $u = f_{\mathrm{Q}}(v)$. Then $w = f_{\mathrm{Q}}^{\mathfrak{gv}}(v) \in^{\mathfrak{gv}} f_{\mathrm{P}}^{\mathfrak{gv}}$. If $w \in \omega$ it is easily shown that $w = u$, contradicting $u \notin \omega$. Therefore $w \notin \omega$, and so $w \in^{\mathfrak{gv}} y$. By $\phi(y)$, $w \in y$, whence $\theta(w)$ implies $u = w \in y$. As $u \notin \omega$, $u \in f_{\mathrm{Q}}(y)$.

If instead $u \in f_{\mathrm{Q}}(y)$, $u \in y$. Thus $u \in^{\mathfrak{gv}} y$, whence $u \in^{\mathfrak{gv}} f_{\mathrm{P}}^{\mathfrak{gv}}$. Since $\mathrm{rk}(u) < \mathrm{rk}(y)$, we may use $\psi(u)$ to show $u \in f_{\mathrm{Q}}^{\mathfrak{gv}}(y)$. This proves $\sigma(y)$, and concludes our proof.

□

4.6 Subtheories of amphi-ZF and ZF

Since the translations underlying $\mathfrak{v}$ and $\mathfrak{g}$ apply to any set of axioms in the respective languages, it is interesting to briefly consider variations on the theories ZF and amphi-ZF.

4.6.1 Hereditarily finite sets and amphisets

Of particular interest in combinatorial game theory are the so-called short (i.e. hereditarily finite) games. We can consider such a theory by negating our Infinity axiom (which we call Inf in the following discussion). However some care is needed when looking at

interpretations. For example, when the infinity axiom of ZF (which we call Inf also) is negated, induction can become problematic[1]. Denote each of ZF, amphi-ZF with Infinity negated by ZF − Inf, ZF_2 − Inf respectively. If ZF − Inf* and ZF_2 − Inf* denote the respective theories with an appropriate axiom of transitive containment[2], say TC, we can restrict the above interpretations to achieve the following.

Theorem 4.6.1. $\mathfrak{g}\colon \mathrm{ZF} - \mathrm{Inf}^* \to \mathrm{ZF}_2 - \mathrm{Inf}^*$ and $\mathfrak{v}\colon \mathrm{ZF}_2 - \mathrm{Inf}^* \to \mathrm{ZF} - \mathrm{Inf}^*$ are inverse to one another in INT.

Notice that by a result of Kaye and Wong [22] this implies $\mathrm{ZF}_2 - \mathrm{Inf}^* \cong \mathrm{PA}$ in INT, where PA is the theory of Peano Arithmetic.

By considering models of the theory ZF − Inf − TC (i.e. ZF − Inf with the transitive containment axiom above negated) we can also construct models of ZF_2 − Inf − TC using our interpretations. Thus $\mathrm{ZF}_2 - \mathrm{Inf} \nvdash \mathrm{TC}$.

Since the definition of our interpretation $\mathfrak{g}\colon \mathrm{ZF} \to \mathrm{ZF}_2$ depended so heavily on $\in^{\mathrm{L}}_{\mathrm{R}}$-induction it is possible that an analogous morphism may not exist between ZF − Inf and ZF_2 − Inf. While the relative translation associated with $\mathfrak{v}$ is obviously definable in all models of ZF − Inf, we required some induction to prove $\mathrm{Found}^{\mathfrak{v}}$. For these reasons it seems highly unlikely that the two theories ZF − Inf and ZF_2 − Inf could be synonymous. However the underlying translations of $\mathfrak{l}$ and $\mathfrak{v}$ do give us interpretations $\mathfrak{l}'$ and $\mathfrak{v}'$ as follows, where $\mathrm{ZF}_2{}^-$ denotes the theory of amphi-ZF without the axiom of foundation.

$$\mathfrak{l}'\colon \mathrm{ZF} - \mathrm{Inf} \to \mathrm{ZF}_2 - \mathrm{Inf};$$
$$\mathfrak{v}'\colon \mathrm{ZF}_2{}^- - \mathrm{Inf} \to \mathrm{ZF} - \mathrm{Inf}\,.$$

It should be clear that from $\mathfrak{v}'$ we can show ZF − Inf interprets ZF_2 − Inf, by restricting to the wellfounded sets; hence these theories have equal consistency strength, even if they are not synonymous. Whether or not the above mentioned theories are subject to other forms of isomorphism (as discussed by Visser [34, p.14]) may be of interest.

[1] In particular ZF − Inf does not prove transitive closure, or equivalently $\in$-induction [22].

[2] In ZF we take $\forall x\, \exists y(x \subseteq y \land \forall u\, \forall v(u \in v \land v \in y \to u \in y))$; in amphi-ZF we take the same, but with $\in$ replaced by $\in^{\mathrm{L}}_{\mathrm{R}}$.

CHAPTER 5

NONSTANDARD TOPOLOGICAL SET THEORY

We turn our attention to the construction of a universe of sets having interesting topological properties. Specifically, we construct in a nonstandard model $\mathcal{U}$ of ZFC, minus the axioms of infinity, power set and foundation, a bisimulation $\sim$; the quotient $\mathcal{U}/{\sim}$ is both a (generalised) metric space and a model of some weak set theory. Such facts hint at similarity with, for example, the κ-hyperuniverses of Forti and Honsell [14, 15] and Forti and Hinnion [12]. Indeed, although the construction and intended use differs, our structure resembles a nonstandard analogue of such spaces; therefore some comparison has been included.

5.1 A base theory of sets

Let EST be the set theory with axioms of extensionality (Ext), zero set (Zero), pair set (Pair), union, (Union), along with the replacement scheme (Rep). Essentially EST is $\mathrm{ZF}^- - \mathrm{Pow} - \mathrm{Inf}$, where ZF^- denotes ZF without the axiom of foundation, Pow the power set axiom, and Inf that of infinity[1]. In this section we will briefly concern ourselves with showing that EST, with one addition, is powerful enough to work in. It may be possible that we can weaken our assumptions further, say by only requiring replacement for functions definable by formulas restricted to a particular class.

In what follows an ordinal recursion scheme is fundamental; we first prove that one such scheme follows from EST. However, as we are considering fragments of ZF we prefer to be explicit, since many notions which are equivalent or trivial when granted the power set and infinity axioms can become complicated. For instance, despite giving us ordinal-induction, full $\in$-induction (which is an equivalent theorem over stronger fragments of ZF) does not follow from EST (see Mancini and Zambella [28], or Kaye and Wong [22]).

Definition 5.1.1. Let $\mathcal{U} \models \mathrm{EST}$. Let $\mathrm{totord}_\in(x)$ be the first-order statement "x is totally ordered by $\in$", and also define formulas $\mathrm{wf}_\in(x)$, $\mathrm{trans}_\in(x)$ by

$$\begin{aligned} \mathrm{wf}_\in(x) &\equiv_{\mathrm{def}} \forall y \subseteq x \; \big(y \neq \varnothing \to \exists z \in y \; \forall w \in y \, (w \notin z)\big); \\ \mathrm{trans}_\in(x) &\equiv_{\mathrm{def}} \forall y \in x \, (y \subseteq x). \end{aligned}$$

[1] Given the replacement scheme we can drop the redundant separation scheme.

We call $\alpha \in \mathcal{U}$ an *ordinal* if and only if

$$\mathcal{U} \vDash \mathrm{wf}_{\in}(\alpha) \wedge \mathrm{totord}_{\in}(\alpha) \wedge \mathrm{trans}_{\in}(\alpha),$$

and use $\mathbf{On}(\mathcal{U})$, or $\mathbf{On}$ when confusion is unlikely, to denote the class of ordinals in $\mathcal{U}$.

We use $\mathrm{Ind}(\mathbf{On})$ to denote a scheme of ordinal-induction, i.e.

$$\forall \overline{a}\ \forall \alpha \in \mathbf{On}\ \big(\forall \beta \in \alpha\ \phi(\beta, \overline{a}) \to \phi(\alpha, \overline{a})\big) \to \forall \alpha \in \mathbf{On}\ \phi(\alpha, \overline{a}), \tag{5.1}$$

for all first-order formulas $\phi(\alpha, \overline{a})$.

Since we are restricting attention to the wellfounded ordinals, we can prove the ordinal-induction scheme from EST. First we require the following lemma, whose proof is taken from Kunen [23]. This proof, and that of the following proposition, are standard; however to reassure that $\mathrm{Ind}(\mathbf{On})$ does indeed follow from EST, we recount them here. If $(A, \leq)$ is a partially ordered set then we denote by $\mathrm{pred}(A, a)$ the class

$$\{a' \in A \colon a' \leq a\}.$$

Lemma 5.1.2. Let $\mathcal{U} \vDash \mathrm{EST}$, and suppose $(A, <), (B, <) \in \mathcal{U}$ are wellordered sets. Then precisely one of the following holds.

- An initial segment of A is isomorphic to B;
- an initial segment of B is isomorphic to A;
- $(A, <) \cong (B, <)$.

Proof. By replacement $A \times B$ is a set (consider $\bigcup\{\{a\} \times B \colon a \in A\}$); therefore

$$f = \{(v, w) \colon v \in A \wedge w \in B \wedge (\mathrm{pred}(A, v), <) \cong (\mathrm{pred}(B, w), <)\}$$

is also a set. We claim f is a function. Indeed, let $a \in A$ be least such that

$$\exists b, b' \in B(b \neq b' \wedge (a, b) \in f \wedge (a, b') \in f).$$

Pick isomorphisms $g \colon \mathrm{pred}(A, a) \to \mathrm{pred}(B, b)$, $h \colon \mathrm{pred}(B, b') \to \mathrm{pred}(A, a)$. Then $g \circ h \colon \mathrm{pred}(B, b') \cong \mathrm{pred}(B, b)$. Suppose, without loss of generality, that $b < b'$, and let $a' \in A$ be the least element not in $\mathrm{im}(h \circ g \circ h)$. Notice that $k = h{\restriction}\mathrm{pred}(B, b)$ is also an isomorphism $\mathrm{pred}(B, b) \to \mathrm{pred}(A, a')$, by choice of a'. We also have that $(k \circ g \circ h)^{-1} \circ k \colon \mathrm{pred}(A, a') \cong \mathrm{pred}(B, b')$, contradicting the leastness of a. Therefore f is a function.

To see f is injective we proceed similarly. Let $a \in A$ be least such that $\exists a' \in A(a' \neq a \wedge f(a) = f(a'))$. Then $f{\restriction}\mathrm{pred}(A, a')$ is an isomorphism $\mathrm{pred}(A, a') \to \mathrm{pred}(B, f(a))$; but also $f{\restriction}\mathrm{pred}(A, a) \colon \mathrm{pred}(A, a) \cong \mathrm{pred}(B, f(a))$. As $a < a'$, similar reasoning to the above gives a contradiction.

Finally, if $\mathrm{dom}(f) \subseteq A$ and $\mathrm{im}(f) \subseteq B$, take $a \in A, b \in B$ least such that $a \notin \mathrm{dom}(f)$ and $b \notin \mathrm{im}(f)$. Then $\mathrm{pred}(A, a) \cong \mathrm{pred}(B, b)$, so it follows that $(a, b) \in f$, a contradiction. Hence either f or f^{-1} is an embedding. Notice also that, since each order is total and f, f^{-1} are order embeddings, $\mathrm{dom}(f), \mathrm{im}(f)$ are necessarily initial segments. $\square$

Now we are able to prove that $\mathrm{EST} \vdash \mathrm{Ind}(\mathbf{On})$.

Proposition 5.1.3. The ordinal-induction scheme Ind($\mathbf{On}$) follows from EST.

Proof. Let $\mathcal{U} \models$ EST, and take a nonempty subclass $\mathcal{A}$ of $\mathbf{On}(\mathcal{U})$. In the usual way we can show $\mathcal{A}$ has a least element: pick $\alpha \in \mathcal{A}$. If $\alpha \cap \mathcal{A}$ is empty, α is minimal in $\mathcal{A}$, and the lemma shows that, indeed, α is least in $\mathcal{A}$. Otherwise, as $\mathcal{U} \models \mathrm{wf}_{\in}(\alpha)$, $\alpha \cap \mathcal{A}$ contains an $\in$-minimal element β. Take any $\gamma \in \mathcal{A}$. If $\gamma \in \alpha$, $\beta \in \gamma$ or $\beta = \gamma$. If $\gamma \notin \alpha$, by trichotomy $\alpha = \gamma$ or $\alpha \in \gamma$, so $\beta \in \gamma$. Ordinal induction then follows for a formula $\phi(\alpha, \overline{a})$ by considering the least α such that $\neg\phi(\alpha, \overline{a})$. □

We can then prove an ordinal recursion theorem, which we state as follows.

Proposition 5.1.4. Assume $\mathcal{U} \models$ EST. If $F \colon \mathcal{U} \times \mathbf{On} \to \mathcal{U}$ is a class function then there is a unique class function $G \colon \mathbf{On} \to \mathcal{U}$ such that

$$\mathcal{U} \models \forall \alpha \in \mathbf{On}(G(\alpha) = F(G{\restriction}\alpha)).$$

Proof. As usual, call a set function f an attempt if $\mathrm{dom}(f) \in \mathbf{On}$ and for all $\beta \in \mathrm{dom}(f)$, $f(\beta) = F(f{\restriction}\beta, \beta)$. Suppose f, g are attempts; let $\delta = \mathrm{dom}(f) \cap \mathrm{dom}(g)$. If $\beta \in \delta$ and $f{\restriction}\beta = g{\restriction}\beta$ then $f(\beta) = G(f{\restriction}\beta, \beta) = g(\beta)$. By induction, then, $f{\restriction}\delta = g{\restriction}\delta$.

Let $\alpha \in \mathbf{On}$ and suppose that for all $\beta \in \alpha$ there is an attempt with domain α. Since an attempt is uniquely defined by its domain, we can define f_β to be the attempt having domain β, for all $\beta \in \alpha$. By replacement $\{f_\beta \colon \beta \in \alpha\}$ is a set, and so $g = \bigcup_{\beta\in\alpha}$ exists in $\mathcal{U}$. If $z = F(g, \alpha)$ then $g \cup \{(\alpha, z)\}$ is an attempt whose domain contains α as a member. By induction there is a unique attempt having domain α, for all $\alpha \in \mathbf{On}$. Define $G(\gamma)$ to be the unique w such that for every attempt f, $\alpha \in \mathrm{dom}(f) \to f(\alpha) = w$. □

In addition to ordinal-induction, we will require the axiom of choice, AC; this is necessary for our construction, although a more general method can be used to avoid AC if desired. In the remainder of this chapter $\mathcal{U}$ will be a model of the set theory EST + AC.

5.2 Two Pseudometrics on nonstandard sets

We begin by introducing some notation and terminology. By $\mathbb{R}_e$ we denote the set $[-\infty, \infty]$. For a nonstandard model ${}^*\mathbb{R}_e$, we define the standard part map, denoted st or $^\circ$, as usual: if $x \in {}^*\mathbb{R}_e$ differs from a standard real number r by an infinitesimal then $\mathrm{st}(x) = r$; otherwise $\mathrm{st}(x)$ is ∞ or $-\infty$, depending on the sign of x.

We use $\mathbb{N}$ for the set of natural numbers ($\{0, 1, 2, \ldots\}$) and $\mathbb{N}^+$ for the set of positive natural numbers, $\{1, 2, 3, \ldots\}$. A pseudometric on a space X is taken to be a function $\tau \colon X \times X \to [0, \infty]$ which satisfies

- $\forall x\, (\tau(x, x) = 0)$;
- $\forall x, y\, (\tau(x, y) = \tau(y, x))$;
- $\forall x, y, z\, (\tau(x, z) \leq \tau(x, y) + \tau(y, z))$.

Notice in particular that we allow infinite distances. Such a function will always be equivalent to a true, finite-valued pseudometric. For instance, we can define $\tau'(x,y) = \tau(x,y)/(1+\tau(x,y))$. A pseudo-ultrametric is a pseudometric τ which satisfies the stronger triangle inequality,

$$\forall x,y,z\,(\tau(x,z) \leq \max(\tau(x,y),\tau(y,z))).$$

If τ satisfies $\forall x,y\,(\tau(x,y) = 0 \to x = y)$ then we drop the prefix pseudo, obtaining a metric or ultrametric. We occasionally refer to pseudometrics or *pseudometrics simply as *distances*.

Henceforth we fix a nonstandard model $\mathcal{M} = ({}^*\mathcal{U}, {}^*\mathbb{R}_e, d, \ldots)$, where d is a *pseudometric on ${}^*\mathcal{U}$, taking values in ${}^*\mathbb{R}_e$. No restrictions are placed on this pseudometric; we could, for example, take the trivial function which is identically zero.

5.2.1 Links and chains

The fundamental objects we consider here are links and chains. Links are recursively defined mappings between sets, which allow us to construct a pseudometric based on d. Informally, if x, y are sets in $\mathcal{U}$ we can view the disjoint union $x \uplus y$ as a bipartite graph G (in the obvious way) by separating the elements of x from those of y; then a link $p\colon x \circledcirc y$ is similar to a weighted matching of these two parts[1]. A chain is then a sequence of composed links.

Throughout we will denote a sequence of elements a_i by $\langle a_0, \ldots, a_m\rangle$, where a_m is the last element, or simply by $\langle a_i\rangle$ when m is understood from the context. For two sequences $\langle a_0, \ldots, a_m\rangle$ and $\langle b_0, \ldots, b_n\rangle$ we let $\langle a_i\rangle \frown \langle b_i\rangle$ denote their concatenation, $\langle a_0, \ldots, a_m, b_0, \ldots, b_n\rangle$.

Definition 5.2.1. Let $x, y \in {}^*\mathcal{U}$. The 0-*link* from x to y is the quadruple $\mathrm{zero}(x,y) = (x, y, \varnothing, \varnothing)$. The *weight* of $\mathrm{zero}(x,y)$ is defined to be $\mathrm{w}(\mathrm{zero}(x,y)) = d(x,y)$. In general if p links x to y we write $p\colon x \circledcirc y$. We call x the *source* of p (denoted $\operatorname{src} p$) and y the *target* of p (written $\operatorname{trg} p$).

Assuming k-links have been defined for all $k \leq n$, an n-*chain* from x to y is a pair $a = (s^a, l^a)$, such that:

- $s^a = \langle s^a_0 \ldots s^a_k\rangle$ is a *finite sequence of sets $s^a_i \in {}^*\mathcal{U}$, with $s^a_0 = x$ and $s^a_k = y$;
- $l^a = \langle l^a_0 \ldots l^a_{k-1}\rangle$ is a *finite sequence of links $l^a_i\colon s^a_i \circledcirc s^a_{i+1}$;
- each l^a_i is a k_i-link for some $k_i \leq n$.

The weight of a is then $\mathrm{w}(a) = \sum_{i=0}^{k-1} \mathrm{w}(l^a_i)$. In general if a is a chain from x to y we write $a\colon x \rightsquigarrow y$. As with links, we call x, y respectively the *source* and *target!of a chain*, and write $x = \operatorname{trg} a,\ y = \operatorname{src} a$.

Assuming k-chains have been defined for all $k < n$, an n-link from x to y is a tuple $p = (x, y, p_x, p_y)$, where p_x, p_y are functions with respective domains x, y, and which satisfy the following.

[1]Technically a matching involves a bijection between two parts of a bipartite graph, whereas a link involves a pair of functions in opposite directions, which typically are not even injective.

- For all $u \in x$, $p_x(u)$ is an m_u-chain, for some $m_u < n$, with $\operatorname{src} p_x(u) = u$ and $\operatorname{trg} p_x(u) \in y$.
- for all $v \in y$, $p_y(v)$ is an m_v-chain, for some $m_v < n$, with $\operatorname{src} p_y(v) = v$ and $\operatorname{trg} p_y(v) \in x$.

The weight of p is $\mathrm{w}(p) = \sup\{\mathrm{w}(p_x(u)) \colon u \in x\} \cup \{\mathrm{w}(p_y(v)) \colon v \in y\}$.

Often we will drop the prefix natural number and refer simply to *links* and *chains*.

The following simple example demonstrates how chains and links are constructed.

Example 5.2.2. Let $x = 3 = \{0, 1, 2\}$ and $y = 2 = \{0, 1\}$. In the discussion below, links will be weighted according to the pseudometric d; hence, in accordance with intuition, we can always find an "optimal" link which chains 0 to 0 and 1 to 1 (notice that if d is degenerate then not all optimal links will necessarily fix $0, 1$). Thus we only concern ourselves with producing a chain a from $2 \in x$ to either of $0, 1 \in y$. There are three obvious such links. The first two concern the simplest chain possible from 2 to either of $0, 1$: simply take the zero link $p \colon 2 \circledcirc 0$, say, and then form the chain $a \colon 2 \rightsquigarrow 0$ for which s^a is the two-element sequence $\langle 2, 0\rangle$, and l^a is the single-element sequence $\langle p\rangle$. Similarly we could link $2 \in x$ to $1 \in y$.

Instead we could define a chain $b \colon 2 \rightsquigarrow 1$ as follows. Let q be the link $2 \circledcirc 1$ with chains $\mathrm{zero}(0, 0)$ and $\mathrm{zero}(1, 0)$. Take $s^b = \langle 2, 1\rangle$ and $l^b = \langle q\rangle$. Notice $\mathrm{w}(b) \leq \mathrm{w}(a)$ regardless of d, and that b is in a sense deeper than a.

Note that in general, given any link $p \colon x \circledcirc y$, we can construct a chain $x \rightsquigarrow y$ as the pair $(\langle x, y\rangle, \langle p\rangle)$. Since this will arise frequently below, we denote the chain so constructed by $\hat{p}$.

In Example 5.2.2 we alluded to an intuitive notion of *depth*; the following definition formalises this notion.

Definition 5.2.3. The *lower hereditary depth* function, denoted $\lfloor\cdot\rfloor$, is defined by setting

$$\lfloor \mathrm{zero}(x, y)\rfloor = \begin{cases} 0 \text{ if } x \neq y; \\ \infty \text{ if } x = y \end{cases}$$

and then

- $\lfloor a\rfloor = \min_i \lfloor l_i^a\rfloor$ for all chains a;
- $\lfloor p\rfloor = \min\big(\{\lfloor p_x(u)\rfloor \colon u \in x\} \cup \{\lfloor p_y(v)\rfloor \colon v \in y\}\big) + 1$ for non-zero links p.

The *upper hereditary depth* function, denoted by $\lceil\cdot\rceil$, is defined similarly:

$$\lceil \mathrm{zero}(x, y)\rceil = \begin{cases} 0 \text{ if } x = y; \\ \infty \text{ if } x \neq y \end{cases}$$

and

- $\lceil a\rceil = \max_i \lceil l_i^a\rceil$ for all chains a;
- $\lceil p\rceil = \max\big(\{\lceil p_x(u)\rceil \colon u \in x\} \cup \{\lceil p_y(v)\rceil \colon v \in y\}\big) + 1$ for non-zero links p.

Remarks 5.2.4.

- Notice that above we defined $\lfloor\rfloor$, $\lceil\rceil$ for links by taking the maximum and minimum of values ranging over a possibly *infinite set. This is certainly fine in the case of $\lfloor\rfloor$, taking minimums, since every internal set has a least ordinal and each value is a *natural number. In the case of $\lceil\rceil$, the maximum also exists: recall that each link p is a k-link for some k, and hence the set

$$D = \{\lceil p_x(u)\rceil, \lceil p_y(v)\rceil : x \in x, v \in y\}$$

is bounded. Therefore the maximum exists and is equal to $\min\{k \in {}^*\mathbb{N} : \forall x \in D\,(k \geq n)\}$.

- We can envisage $\lfloor\cdot\rfloor, \lceil\cdot\rceil$ as the least and greatest possible times (or amount of some other resource) taken to complete a specific game, with one player adhering to a strategy, while the weight function w quantifies some notion of *cost* of the strategy. See section 5.5 for more details.

A link $p\colon x \circledcirc y$ simply matches the elements of x with those of y, such that every element is matched (hence the injectivity requirement must be dropped; therefore links are not, in general, matchings of bipartite graphs). The situation is complicated by the need to make these matchings deep, in the sense that the first ω levels of matched sets' $\in$-trees are isomorphic. Specifically we will consider links with infinite lower hereditary depths. Furthermore, in order to obtain some sort of distance function from these matchings we must be able to compose chains; hence their length cannot be bounded.

Claim 5.2.5. Suppose $a\colon x \rightsquigarrow y$, $b\colon y \rightsquigarrow z$, $p\colon x \circledcirc y$ and $q\colon y \circledcirc z$. Then

- there is a chain $a'\colon y \rightsquigarrow x$ of weight and hereditary depths equal to those of a;

- there is a link $p'\colon y \circledcirc x$ of weight and hereditary depths equal to those of p;

- there is a composite chain $ab\colon x \rightsquigarrow z$ with $\lceil ab\rceil = \max(\lceil a\rceil, \lceil b\rceil)$, $\lfloor ab\rfloor = \min(\lfloor a\rfloor, \lfloor b\rfloor)$ and $\mathrm{w}(ab) = \mathrm{w}(a) + \mathrm{w}(b)$;

- if $\lfloor p\rfloor, \lfloor q\rfloor > 0$ then there is a composite link $pq\colon x \circledcirc z$ with $\lceil pq\rceil = \max(\lceil p\rceil, \lceil q\rceil)$, $\lfloor pq\rfloor = \min(\lfloor p\rfloor, \lfloor q\rfloor)$ and $\mathrm{w}(pq) \leq \mathrm{w}(p) + \mathrm{w}(q)$.

Proof. The first two claims are easily proved. If $a\colon x \rightsquigarrow y$ and $b\colon y \rightsquigarrow z$ then we can clearly compose a with b: assuming $s^a = \langle s_0 \ldots s_m\rangle$ and $s^b = \langle s_0 \ldots s_n\rangle$, take $s^{ab} = \langle s^a_0 \ldots s^a_m s^b_1 \ldots s^b_n\rangle$ (notice we have avoided repetition of $y = s^b_0$ in this sequence), and we take $l^{ab} = l^a \frown l^b$ to be the concatenation of the sequences l^a and l^b. Clearly this defines a chain $ab\colon x \rightsquigarrow z$, with $\mathrm{w}(ab) = \mathrm{w}(a) + \mathrm{w}(b)$, $\lfloor ab\rfloor = \min\{\lfloor a\rfloor, \lfloor b\rfloor\}$ and $\lceil ab\rceil = \max\{\lceil a\rceil, \lceil b\rceil\}$.

We can also compose links as follows, provided they have positive lower hereditary depth. First, suppose that one of p, q is a zero link. Then in particular $x = y$ or $y = z$; hence we define the composite to be q or p respectively, which clearly satisfies the desired criteria. Now suppose that p, q are non-zero. Then whenever $u \in x$ there is a chain $p_x(u)\colon u \rightsquigarrow v$ for some $v \in y$; but then $q_y(v)\colon v \rightsquigarrow w$ for some $w \in z$. Define $(pq)_x(u)$ to be $p_x(u)q_y(v)$. Then $\mathrm{w}((pq)_x(u)) = \mathrm{w}(p_x(u)) + \mathrm{w}(q_y(v))$, $\lfloor (pq)_x(u)\rfloor = \min\{\lfloor p_x(u)\rfloor, \lfloor q_y(v)\rfloor\}$,

and $\lceil (pq)_x(u) \rceil = \max\{\lceil p_x(u) \rceil, \lceil q_y(v) \rceil\}$. Since v, w are necessarily unique for fixed u, we can define $(pq)_x$ by $(pq)_x(u) = p_x(u)q_y(v)$ in this way. Symmetrically if $w \in z$ then there is $v \in y$ such that $q_z(w)\colon w \rightsquigarrow v$, and so there exists $u \in x$ such that $p_y(v)\colon v \rightsquigarrow u$; therefore we can define $(pq)_z(w)$ to be the composed chain $q_z(w)p_y(v)$. This determines a link $pq = (x, z, (pq)_x, (pq)_z)\colon x \circledcirc z$. Notice that, for $u \in x$ and $w \in z$, we have

$$\mathrm{w}((pq)_x(u)) = \mathrm{w}(p_x(u)) + \mathrm{w}(q_y(\operatorname{trg} p_x(u))) \le \mathrm{w}(p) + \mathrm{w}(q); \text{ and}$$
$$\mathrm{w}((pq)_z(w)) = \mathrm{w}(q_z(w)) + \mathrm{w}(p_y(\operatorname{trg} q_z(w))) \le \mathrm{w}(p) + \mathrm{w}(q).$$

Therefore $\mathrm{w}(pq) \le \mathrm{w}(p)+\mathrm{w}(q)$; it is similarly shown that $\lceil pq \rceil$, $\lfloor pq \rfloor$ are equal to $\max\{\lceil p \rceil, \lceil q \rceil\}$ and $\min\{\lfloor p \rfloor, \lfloor q \rfloor\}$ respectively. □

Suppose $a\colon x \rightsquigarrow y$ and $\lfloor a \rfloor > 0$. Then we can define a link $\breve{a}\colon x \circledcirc y$ as the composite $l_0^a \dots l_{k-1}^a$ of all the links in a. Notice that we then have $\breve{\hat{p}} = p$ for all links p, but even where $\breve{a}$ is defined, $\hat{\breve{a}}$ may be different from a.

5.2.2 Two pseudometrics

Suppose $n \in {}^*\mathbb{N}$, $\delta \in {}^*\mathbb{R}_e = {}^*[-\infty, \infty]$ and $x, y \in {}^*\mathcal{U}$. By $\mathrm{ch}(x, y; i, \delta)$ we denote the ${}^*\mathcal{U}$-definable class $\{a\colon \lfloor a \rfloor > i,\ \mathrm{w}(a) < \delta,\ a\colon x \rightsquigarrow y\}$. We will also use $\mathrm{Ch}_\delta(x, y)$ to denote the external class $\bigcap_{i \in \mathbb{N}^+} \mathrm{ch}(x, y; i, \delta)$.

Definition 5.2.6. We define two pseudometrics as follows.

- For $x, y \in {}^*\mathcal{U}$, define

  $$\sigma(x, y) = \inf\{{}^\circ\mathrm{w}(a)\colon a \in \mathrm{Ch}_\infty(x, y)\}$$

  otherwise.

- Fix an infinitesimal $\alpha > 0$. For each non-infinitesimal $\delta > 0$, define an external function ρ_δ by

  $$\rho_\delta(x, y) = \inf\{{}^\circ(\alpha\lceil a \rceil)\colon a \in \mathrm{Ch}_\delta(x, y)\}.$$

  Since for $\delta_1 \le \delta_2$ we have $\rho_{\delta_1}(x, y) \ge \rho_{\delta_2}(x, y)$, we can define a function ρ as the pointwise limit $\rho = \lim_{\delta \to 0^+} \rho_\delta$, taking values in $[0, \infty]$.

Remark 5.2.7. The class $\mathrm{Ch}_\delta(x, y)$ can be replaced by

$$\mathrm{Ch}_\delta^I(x, y) = \bigcap_{i \in I} \mathrm{ch}(x, y; i, \delta)$$

for any cut $I \subseteq_e {}^*\mathbb{N}$. Then each of the functions defined above is still a pseudometric, and moreover many of the results which follow still apply.

It is worthwhile to have a simple characterisation of these functions, as is often possible in nonstandard constructions. From now on a link p (respectively, a chain a) will be called *deep* if $\lfloor p \rfloor > \mathbb{N}$ ($\lfloor a \rfloor > \mathbb{N}$).

Proposition 5.2.8. Let $x, y \in {}^*\mathcal{U}$.

- If $\sigma(x,y) < \infty$ there is a deep chain $a\colon x \circledast y$ such that $\mathrm{w}(a) \approx \sigma(x,y)$.
- If $\rho(x,y) < \infty$ there is a deep chain $a\colon x \circledast y$ with $\mathrm{w}(a) \approx 0$ and $\alpha\lceil a \rceil \approx \rho(x,y)$.

Proof. Consider the formulas

$$\begin{aligned}\phi_\sigma(a,x,y,n,\delta) &\equiv a\colon x \rightsquigarrow y \wedge \lfloor a \rfloor > n \wedge \mathrm{w}(a) < \delta;\\ \phi_\rho(a,x,y,n,\delta) &\equiv a\colon x \rightsquigarrow y \wedge \lfloor a \rfloor > n \wedge \mathrm{w}(a) < 1/n \wedge \alpha\lceil a \rceil < \delta.\end{aligned}$$

Suppose $\sigma(x,y) < \infty$. Then for all $n \in \mathbb{N}^+$ there exists a deep chain $a\colon x \rightsquigarrow y$ such that $\mathrm{w}(a) < \sigma(x,y) + 1/n$; hence

$${}^*\mathcal{U} \vDash \phi_\sigma(a,x,y,n,\sigma(x,y) + 1/n).$$

By overspill ${}^*\mathcal{U} \vDash \exists a\ \phi_\sigma(a,x,y,n,1/n)$ for some $n > \mathbb{N}$. Since a is deep $\sigma(x,y) \le {}^\circ\mathrm{w}(a)$ and so ${}^\circ\mathrm{w}(a) = \sigma(x,y)$.

If instead we assume $\rho(x,y) < \infty$ then first we note that each $\rho_\varepsilon(x,y) < \infty$. Let $m \in \mathbb{N}^+$ and fix $n > m$ such that $\rho(x,y) - \rho_{1/n}(x,y) < 1/2m$. There is a chain $a\colon x \rightsquigarrow y$ such that $\lfloor a \rfloor > \mathbb{N}$, $w(a) < 1/n < 1/m$, and $|\alpha\lceil a \rceil - \rho_{1/n}(x,y)| < 1/2m$. Therefore

$$\phi_\rho(a,x,y,m,\rho(x,y) + 1/m).$$

Since m was arbitrary, by overspill there is an infinite such m; hence there is a chain $a\colon x \rightsquigarrow y$ with $\mathrm{w}(a) \approx 0$, $\lfloor a \rfloor > \mathbb{N}$, and $\rho(x,y) \le {}^\circ(\alpha\lceil a \rceil) \le \rho(x,y)$. $\square$

Remark 5.2.9.

- Proposition 5.2.8 shows that we could give a simpler definition of ρ:

$$\rho(x,y) = \inf\{{}^\circ(\alpha\lceil a \rceil)\colon \lfloor a \rfloor > \mathbb{N},\ \mathrm{w}(a) \approx 0,\ a\colon x \rightsquigarrow y\}.$$

- If $a\colon x \rightsquigarrow y$ is deep then the link $\breve{a} = l_0^a \dots l_{k-1}^a$, say, has weight $\mathrm{w}(\breve{a}) \le \mathrm{w}(a)$ and depths $\lfloor a \rfloor = \lfloor \breve{a} \rfloor$, $\lceil \breve{a} \rceil = \lceil a \rceil$. If $p\colon x \circledast y$ then $\hat{p} = (\langle x,y \rangle, \langle p \rangle)$ has $\lfloor \hat{p} \rfloor = \lfloor p \rfloor$, $\lceil \hat{p} \rceil = \lceil p \rceil$ and weight $\mathrm{w}(\hat{p}) = \mathrm{w}(p)$. Therefore we can replace the word "chain" with "link" in both Proposition 5.2.8 and in the definitions of ρ, σ. This is a simple yet key observation, which we will make use of below.

It is now fairly simple to prove the following.

Proposition 5.2.10. Both σ, ρ are pseudometrics on ${}^*\mathcal{U}$. In fact, ρ is a *pseudo-ultrametric.*

Proof. In both cases each axiom is clear, except for the relevant triangle inequality. For the first claim, suppose $x, y, z \in {}^*\mathcal{U}$. If two or more of x, y, z are equal, or if one of $\sigma(x,y), \sigma(y,z)$ is infinite, then the inequality is easily shown. Assume that $0 < \sigma(x,y), \sigma(y,z) < \infty$ and take chains $a\colon x \rightsquigarrow y$, $b\colon y \rightsquigarrow z$ such that $\lfloor a \rfloor, \lfloor b \rfloor > \mathbb{N}$ and $\mathrm{w}(a) \approx \sigma(x,y)$, $\mathrm{w}(b) \approx \sigma(y,z)$. Then the composite chain $ab\colon x \rightsquigarrow z$ has weight $\mathrm{w}(ab) = \mathrm{w}(a) + \mathrm{w}(b) \approx \sigma(x,y) + \sigma(y,z)$, and infinite depth. Since $\sigma(x,z) \le {}^\circ\mathrm{w}(ab)$ we are done.

In the case of ρ, again if two or more of x, y, z are equal or if one of the distances on the right hand side is infinite, then we have nothing to prove. Assuming that $0 < \rho(x,y), \rho(y,z) < \infty$, we may take chains $a\colon x \rightsquigarrow y$ and $b\colon y \rightsquigarrow z$, both of which are deep and have infinitesimal weight, along with $\alpha\lceil a \rceil \approx \rho(x,y)$ and $\alpha\lceil b \rceil \approx \rho(y,z)$; then as above we have

$$\begin{aligned} \rho(x,z) &\leq {}^{\circ}(\alpha\lceil ab \rceil) \\ &= {}^{\circ}(\alpha \max(\lceil a \rceil, \lceil b \rceil)) \\ &= \max(\rho(x,y), \rho(y,z)). \end{aligned}$$

□

In the following discussion we will rarely distinguish between ρ and σ; rather, we let τ be either one, and derive analogous results for each.

Define a relation $\sim$ on ${}^*\mathcal{U}$ by $x \sim y$ if and only if $\tau(x,y) = 0$. It is easily shown that, since τ is a pseudometric, $\sim$ is an equivalence relation, and τ induces a metric on the quotient space ${}^*\mathcal{U}/{\sim}$.

There are various ways in which we might define such a quotient space. The simplest is to restrict $\mathrm{Th}(\mathcal{U})$ so that $\mathcal{U}$ is actually a set in our external theory, say ZF, perhaps with some large cardinal axiom. Then clearly the usual definition involving equivalence classes is valid. If working without any large cardinals in our external universe, this would involve dropping either Inf or Pow from $\mathrm{Th}(\mathcal{U})$. Notice that, since ${}^*\mathcal{U}$ may not satisfy foundation we cannot define equivalence classes as sets of equivalent objects of least rank.

Alternatively we can attempt to mimic the method of Forti et al. [14, p.12], by defining a map on $\mathcal{U}$ which picks out a unique representative for each equivalence class. This method, however, would require additional restrictions on the original space $(\mathcal{U}, d)$, since in our case many sets will not be $\mathcal{U}$-definable.

For the following discussion, it is not necessary to make such a choice; we merely assume that some suitable quotient $\widehat{\mathcal{U}} = {}^*\mathcal{U}/{\sim}$ has been chosen.

5.3 Set membership in $\widehat{\mathcal{U}}$

In this section we discuss a notion of set membership on the quotient space $\widehat{\mathcal{U}}$.

Proposition 5.3.1. Let $u, x, y \in {}^*\mathcal{U}$ with $x \sim y$, and suppose $u \in x$. Then there exists $v \in y$ such that $u \sim v$. That is, $\sim$ is a bisimulation[1].

Proof. In light of Proposition 5.2.8 and remark 5.2.9 this is straightforward. For notational ease, define formulas ψ_τ for links, analogous to the formulas ϕ_τ for chains:

$$\begin{aligned} &\psi_\sigma(p, x, y, n, \delta) \equiv p\colon x \circledcirc y \wedge \lfloor p \rfloor > n \wedge \mathrm{w}(p) < \delta; \\ &\psi_\rho(p, x, y, n, \delta) \equiv p\colon x \circledcirc y \wedge \lfloor p \rfloor > n \wedge \mathrm{w}(p) < 1/n \wedge \alpha\lceil p \rceil < \delta. \end{aligned}$$

[1]Specifically, $\sim$ is a bisimulation in the sense of Aczel [2].

Since $x \sim y$ there is a link $p \colon x \circledcirc y$ satisfying $\psi_\tau(p, x, y, n, \delta)$ for some infinite n and infinitesimal δ. Therefore p specifies a chain a from u to some $v \in y$, such that $\phi_\tau(a, u, v, n-1, \delta)$ is true; hence $u \sim v$.

□

Definition 5.3.2. If $u, x \in {}^*\mathcal{U}$ then we write $u \mathrel{\mathtt{E}} x$ if and only if there exists $v \in x$ such that $u \sim v$.

Corollary 5.3.3. If $v \sim u \mathrel{\mathtt{E}} x \sim y$ then $v \mathrel{\mathtt{E}} y$.

Denote $x/{\sim}$ by $\hat{x}$. For $\hat{u}, \hat{x} \in \widehat{\mathcal{U}}$, we write $\hat{u} \mathrel{\mathtt{E}} \hat{x}$ if and only if $u \mathrel{\mathtt{E}} x$ (there is no risk of confusion); by Corollary 5.3.3 this relation is well-defined.

Lemma 5.3.4. Let $x, y \in {}^*\mathcal{U}$ and suppose that

- for all $u \in x$ there exists $v \in y$ such that $\tau(u, v) \leq \varepsilon$; and
- for all $v \in y$ there exists $u \in x$ such that $\tau(u, v) \leq \varepsilon$.

Then $\tau(x, y) \leq \varepsilon$.

Proof. Take τ to be either σ or ρ, and recall the formulas ϕ_τ.

For each $u \in x$ and $n \in {}^*\mathbb{N}^+$, let

$$C(u, n, x) = \{a \colon \exists v \in y\, \phi_\tau(a, u, v, n, \varepsilon + 1/n)\},$$

and analogously we take

$$C(v, n, y) = \{a \colon \exists u \in x\, \phi_\tau(a, v, u, n, \varepsilon + 1/n)\}$$

for $v \in y$. Notice that when $n \in \mathbb{N}^+$, each such set is nonempty. By overspill there exists $n > \mathbb{N}$ such that

$$\forall u \in x\, (C(u, n, x) \neq \varnothing) \wedge \forall v \in x\, (C(v, n, y) \neq \varnothing).$$

Let $\mathscr{C}(z) = \{C(t, n, z) \colon t \in z\}$ for $z = x, y$. By *AC there are functions p_x, p_y with respective domains $\mathscr{C}(x), \mathscr{C}(y)$ such that $p_z(t) \in C(t, n, z)$ for all $u \in t$. Let

$$p = (x, y, p_x, p_y).$$

Then $p \colon x \circledcirc y$, and further,

$${}^*\mathcal{U} \vDash \psi_\tau(p, x, y, n, \varepsilon + 1/n).$$

Therefore by proposition 5.2.8, $\tau(x, y) \leq \varepsilon$. □

Theorem 5.3.5. Suppose that $u \mathrel{\mathtt{E}} x$ exactly when $u \mathrel{\mathtt{E}} y$; then $x \sim y$. Therefore $(\widehat{\mathcal{U}}, \mathtt{E}) \vDash \mathrm{Ext}$.

Proof. Let $u \in x$. Then $u \mathrel{\mathtt{E}} x$, so $u \mathrel{\mathtt{E}} y$. Therefore there exists an element $v \in y$ such that $\tau(u, v) = 0$. Similarly if $v \in y$ then $v \mathrel{\mathtt{E}} x$ so for some $u \in x$, $\tau(u, v) = 0$. By Lemma 5.3.4 we have $\tau(x, y) = 0$. □

Our proof that $(\widehat{\mathcal{U}}, \mathtt{E})$ is extensional is heavily reliant on the axiom of choice (AC) being true in $\mathcal{U}$. In fact the following is true.

Proposition 5.3.6. Assume $\mathcal{U} \vDash$ EST. Then the following statements are equivalent.

- For all *pseudometrics d on ${}^*\mathcal{U}$, $({}^*\mathcal{U}/\sigma, \mathtt{E}) \vDash$ Ext;
- for all *pseudometrics d on ${}^*\mathcal{U}$, $({}^*\mathcal{U}/\rho, \mathtt{E}) \vDash$ Ext;
- $\mathcal{U} \vDash AC$.

We will not prove proposition 5.3.6 here, since the proof is highly technical, without offering any real insight. It should be clear, however, that a link from a set x to its union $\bigcup x$ behaves similarly to a choice function. The main difficulty of proving proposition 5.3.6 is therefore in defining a suitable metric d, and in choosing appropriate sets to link (in particular we must ensure that the link $x \multimap \bigcup x$ is deep in order to use extensionality). Although not very difficult, this is a very technical issue.

We must also remark that avoiding choice altogether is feasible. Instead of links having specific targets, they might permit sets of targets. This would not affect the construction in any obvious way, and concepts such as weight and depth could be easily adjusted.

5.4 The structure of $\widehat{\mathcal{U}}$

In this section we consider the set-theoretic and metric structure of $\widehat{\mathcal{U}}$. We begin by arguing that $(\widehat{\mathcal{U}}, \mathtt{E}, \tau)$ is never trivial.

Proposition 5.4.1. The hereditarily finite (standard) sets of $\mathcal{U}$ embed onto $(\widehat{\mathcal{U}}, \mathtt{E})$.

Proof. If $x, y \in \mathcal{U}$ are wellfounded (in the metatheory) then a chain $a\colon x \rightsquigarrow y$ can be deep only if $x = y$. Therefore the map $x \mapsto \hat{x}$ is an embedding when restricted to the wellfounded sets. □

Recall from remark 5.2.7 that a cut I can be used instead of ω to measure "depth" (that is, we might consider a link p deep if and only if $\lfloor p \rfloor > i$ for all $i \in I$). The argument of proposition 5.4.1 can be used to show that in this case

$$\mathcal{U}_I = \{x \in {}^*\mathcal{U} \colon \ \mathrm{wf}_{\in}(x) \wedge \mathrm{rk}(x) \in I\}$$

embeds onto $\widehat{\mathcal{U}}$. Consequently, $\widehat{\mathcal{U}}$ is never trivial.

5.4.1 Metric structure

As discussed by Forti and Honsell [14], since each element of a space such as $\widehat{\mathcal{U}}$ can be regarded as an $\mathtt{E}$-subset of $\widehat{\mathcal{U}}$, it is reasonable to consider these elements with respect to the topology on $\widehat{\mathcal{U}}$. They show that, as a consequence of the free construction principle, it is possible to consider a transitive set N such that $\in_\kappa$ (to which our $\mathtt{E}$ is analogous)

and true membership $\in$ coincide on N[14, p.11]. In our case, the absence of a guaranteed such set forces us to consider instead sets which behave like

$$\tilde{x} = \{\hat{u} \in \widehat{\mathcal{U}} \colon \hat{u} \mathrel{\mathtt{E}} \hat{x}\};$$

as discussed above we assume such objects exist in our external universe.

Recall that for a pseudometric space (X, p) the Hausdorff pseudometric on $\mathscr{P}(X)$ is defined by

$$\mathrm{H}\, p(A, B) = \max\left(\sup_{a \in A} \inf_{b \in B} p(a, b),\ \sup_{b \in B} \inf_{a \in A} p(b, a)\right).$$

If p is nondegenerate (i.e. $p(x, y) = 0$ only when $x = y$) then $\mathrm{H}\, p$ is nondegenerate on $\mathcal{P}_{\mathrm{cl}}(X) = \{A \subseteq X \colon A \text{ is closed}\}$. In fact, $\mathrm{H}\, p(A, B) = 0$ if and only if A and B have the same closure, for $A, B \subseteq X$. Indeed, if $\mathrm{H}\, p(A, B) \neq 0$ then (without loss of generality) there is $a \in A$ such that $\inf_{b \in B} p(a, b) > 0$, so $a \notin \overline{B}$; thus $\overline{A} \neq \overline{B}$; this argument reverses.

The topology on N described by Forti et al. is uniform, and moreover it is shown that this topology admits a κ-ultrametric (called a κ-hypermetric in their papers). This metric, d, is equivalent (but not necessarily equal) to $\mathrm{H}\, d$ on N (note that N contains all of its closed subsets), and so the respective uniformities coincide [14, p.13]. We now turn to the corresponding situation in $\widehat{\mathcal{U}}$. For $x, y \in \widehat{\mathcal{U}}$, let

$$\mathrm{H}_{\mathtt{E}}\tau(\hat{x}, \hat{y}) = \max\big(\sup_{u \mathtt{E} x} \inf_{v \mathtt{E} y} \tau(u, v),\ \inf_{v \mathtt{E} y} \sup_{u \mathtt{E} x} \tau(v, u)\big).$$

Proposition 5.4.2. For all $x, y \in \widehat{\mathcal{U}}$, $\mathrm{H}_{\mathtt{E}}\tau(x, y)$, $\mathrm{H}\, \tau(\tilde{x}, \tilde{y})$, $\mathrm{H}\, \tau(x, y)$ and $\tau(x, y)$ are equal.

Proof. Since the statements "$u \in \tilde{x}$", "$u \mathrel{\mathtt{E}} x$" and "$u \sim v$ for some $v \in x$" are equivalent, $\mathrm{H}_{\mathtt{E}}\tau(x, y)$ is equal to both $\mathrm{H}\, \tau(x, y)$ and $\mathrm{H}\, \tau(\tilde{x}, \tilde{y})$ are equal, for all $x, y \in {}^*\mathcal{U}$.

Lemma 5.3.4 shows that $\tau(x, y) \leq \mathrm{H}_{\mathtt{E}}\tau(x, y)$ for all x, y. Fix $x, y \in {}^*\mathcal{U}$, and consider $\tau = \sigma$. Let $t = \sigma(x, y)$. If $t = \infty$, $\mathrm{H}_{\mathtt{E}}\tau(x, y)$ is also infinite. If instead $t < \infty$, by proposition 5.2.8 there is a deep link $p \colon x \circledcirc y$ with $\mathrm{w}(p) \approx t$. Therefore for all $u \in x$ there is a $v \in y$ such that $p_x(u) \colon u \rightsquigarrow v$. Moreover $p_x(u)$ is deep, and ${}^\circ\mathrm{w}(p_x(u)) \leq t$. Hence

$$\sup_{u \in x} \inf_{v \in y} \sigma(u, v) \leq t.$$

The analogous statement in the opposite direction is true by the same argument. Hence $\mathrm{H}\, \sigma(x, y) \leq t$.

Consider now the case $\tau = \rho$, and let $t = \rho(x, y)$. There is a deep link $p \colon x \circledcirc y$ such that $\alpha\lceil p \rceil \approx t$ and $\mathrm{w}(p) \approx 0$. Therefore for $u \in x$ there is $v \in y$ with $p_x(u) \colon u \rightsquigarrow v$ satisfying $\mathrm{w}(p_x(u)) \approx 0$, $\alpha\lceil p_x(u) \rceil \approx t$. Since $p_x(u)$ is also deep, $\sup_{u \in x} \inf_{v \in y} \rho(u, v) \leq t$. Again the same is true in the opposite direction, and therefore $\mathrm{H}\, \rho(x, y) = \rho(x, y)$. □

Given that points in this space are themselves sets of points, it is sensible to consider when a set $x \in {}^*\mathcal{U}$ behaves like an open or closed set. That is, when $\{u \colon u \mathrel{\mathtt{E}} x\}$ is a closed subset of ${}^*\mathcal{U}$[1]. That $\mathrm{H}\, \tau$ and τ coincide suggests the following.

[1] Notice that Forti et al. demonstrated that in various hyperuniverses, every set is also closed. In fact this forms part of the definition of *κ-hyperuniverses* [15].

Proposition 5.4.3. For all $u, x \in {}^*\mathcal{U}$,

$$u \mathrel{\mathtt{E}} x \leftrightarrow u \in \overline{x}.$$

Consequently, each class $\tilde{x}$ is closed.

Proof. If $u \mathrel{\mathtt{E}} x$, $u \sim v$ for some $v \in x$, and so trivially $u \in \overline{x}$. If $u \in \overline{x}$, then for each $n \in \mathbb{N}^+$,

$$\exists v \in x \ \exists a \, \phi_\tau(a, u, v, n, 1/n). \qquad (\star)$$

By overspill $(\star)$ applies for some $n > \mathbb{N}$; hence $u \sim v$, and $u \mathrel{\mathtt{E}} x$. □

In a similar vein we have the following. By $(u_n)_{n<k}$ we denote a *finite sequence; by $(u_n)_{n<\omega}$ we denote a possibly external ω-sequence.

Proposition 5.4.4. If $(u_n)_n < k$ is a *finite sequence in ${}^*\mathcal{U}$ such that $(u_n)_{n<\omega}$ is τ-Cauchy, then $(u_n)_{n<\omega}$ has a τ-limit in ${}^*\mathcal{U}$. In particular if ${}^*\mathcal{U}$ is $\aleph_1$-saturated then ${}^*\mathcal{U}$ and $\widehat{\mathcal{U}}$ are complete with respect to τ.

Proof. For all $n \in \mathbb{N}^+$,

$${}^*\mathcal{U} \vDash \exists m \in {}^*\mathbb{N} \ \ \forall i, j \geq m \ \ \exists a \ \phi_\tau(a, u_i, u_j, n, 1/n).$$

Hence by overspill the same statement holds for some infinite $n \in {}^*\mathbb{N}$. We can take $m \in {}^*\mathbb{N}$ such that whenever $i, j \geq m$, $\exists a \ \phi_\tau(a, u_i, u_j, n, 1/n)$, implying $\tau(u_i, u_j) = 0$. In particular u_m is easily seen to be a τ-limit for (u_n). If ${}^*\mathcal{U}$ is $\aleph_1$-saturated then for any given ω-sequence $(x_n)_{n<\omega}$ we can find a *finite extension $(x_n)_{n<k}$ in ${}^*\mathcal{U}$. If $(x_n)_{n<\omega}$ is Cauchy then $(x_n)_{n<\omega}$ has a limit by the above argument. □

5.4.2 Set theory in $\widehat{\mathcal{U}}$

It is interesting to briefly discuss what $\widehat{\mathcal{U}}$ looks like from a set-theoretic perspective.

Proposition 5.4.5. The structure $(\widehat{\mathcal{U}}, \mathtt{E})$ satisfies Zero + Ext + Pair + Union.

Proof. Clearly we never have $v \sim u \in \varnothing$, and so $\hat{\varnothing}$ is $\mathtt{E}$-empty. Theorem 5.3.5 proves Ext. Now suppose that $x, y \in {}^*\mathcal{U}$, and set $z = \{x, y\}$. Then $w \mathrel{\mathtt{E}} z$ if and only if $w \sim x$ or $w \sim y$; hence $\hat{z}$ is the pair-set containing x and y.

Similarly we claim that $\widehat{\bigcup x}$ behaves as a sum-set of x, i.e. for all $z \in {}^*\mathcal{U}$, $z \mathrel{\mathtt{E}} \bigcup x$ if and only if there exists $y \mathrel{\mathtt{E}} x$ such that $z \mathrel{\mathtt{E}} y$. One direction is clear: if $z \mathrel{\mathtt{E}} \bigcup x$ then $z \sim z'$ for some $z \in \bigcup x$, hence $z \sim z' \in y \in x$ for some y. But then $z \mathrel{\mathtt{E}} y \mathrel{\mathtt{E}} x$. Conversely if $z \mathrel{\mathtt{E}} y \mathrel{\mathtt{E}} x$, then $z \sim z' \in y \sim y' \in x$ holds for some $z', y' \in {}^*\mathcal{U}$. By Corollary 5.3.3 $z \mathrel{\mathtt{E}} y' \in x$, hence $z \mathrel{\mathtt{E}} \bigcup x$. □

If the original model $\mathcal{U}$ satisfies stronger set-theoretic axioms, we can show that $\mathrm{Th}(\widehat{\mathcal{U}})$ is also stronger.

Proposition 5.4.6. If $\mathcal{U} \vDash \mathrm{Pow}$ then $\widehat{\mathcal{U}} \vDash \mathrm{Pow}$.

Proof. Let $x \in {}^*\mathcal{U}$. We claim that $z \mathrel{\mathtt{E}} \mathcal{P}x$ precisely when z is an $\mathtt{E}$-subset of x (i.e. whenever $u \mathrel{\mathtt{E}} z$, $u \mathrel{\mathtt{E}} x$). If $z \mathrel{\mathtt{E}} \mathcal{P}x$ then $z \sim y$ for some subset y of x; hence z is trivially a $\mathtt{E}$-subset of x.

Conversely, suppose that whenever $u \mathrel{\mathtt{E}} z$, $u \mathrel{\mathtt{E}} x$. By overspill,

$$\forall u \in z \ \exists v \in x \ \exists a \, \phi_\tau(a, u, v, n, 1/n)$$

for some infinite $n \in {}^*\mathbb{N}$. Using ${}^*\mathrm{AC}$ we can choose a function $f\colon z \to x$ such that

$$\forall u \in z \ \ \exists a \ \ \phi_\tau(a, u, f(u), n, 1/n).$$

But then we can easily find a link $p\colon z \circledcirc \operatorname{im} f$, with $\mathrm{w}(p)$ infinitesimal, $\lfloor p \rfloor > \mathbb{N}$, and if desired, $\alpha\lceil p \rceil \approx 0$ too. □

In the next proposition we take Inf to be the statement, "there exists an inductive set".

Proposition 5.4.7. If $\mathcal{U} \models \mathrm{Inf}$ then $\widehat{\mathcal{U}} \models \mathrm{Inf}$.

Proof. Suppose $(\mathcal{U}, \in) \models \mathrm{Inf}$, and that ω is the minimal inductive set in $\mathcal{U}$. Then ${}^*\omega$ is the minimal inductive set in ${}^*\mathcal{U}$. We have $\varnothing \mathrel{\mathtt{E}} {}^*\omega$, and if $x \mathrel{\mathtt{E}} {}^*\omega$ then $x \sim n$ for some $n \in {}^*\omega$. Therefore $s(x) = x \cup \{x\} \sim s(n) \in {}^*\omega$, so $\widehat{{}^*\omega}$ is inductive. □

We have not proved that $(\widehat{\mathcal{U}}, \mathtt{E})$ is closed under intersection. We cannot expect that $\widehat{x \cap y}$ will always be a candidate for $\hat{x} \cap \hat{y}$. Indeed, if x, y are distinct singletons satisfying $x \sim y$, then $\widehat{x \cap y} = \varnothing$, while $\hat{x} \cap \hat{y}$ is nonempty. We will not investigate any further here, though we can suggest possible solutions.

Firstly we might wish to include an antifoundation axiom (either internally or externally) and proceed as Forti et al. have, finding some transitive set $C \subseteq {}^*\mathcal{U}$ of representatives for the equivalence $\sim$; this would ensure that $\hat{x} \cap \hat{y}$ exists and equals $\widehat{x \cap y}$ for all $x, y \in C$.

A second solution might be to utilise the fact that the Hausdorff distance $\mathrm{H}_{\mathtt{E}}\tau$ coincides with τ; this means $({}^*\mathcal{U}, \tau)$ embeds isometrically onto, for example, $(\mathcal{P}_{\mathrm{cl}}\widehat{\mathcal{U}}, \mathrm{H}'\tau)$. Clearly $\mathcal{P}_{\mathrm{cl}}\widehat{\mathcal{U}}$ is closed under intersections, and by the discussion above $(\mathcal{P}_{\mathrm{cl}}\widehat{\mathcal{U}}, \mathtt{E}')$ is extensional when we define

$$x \mathrel{\mathtt{E}'} y \Leftrightarrow x \in \overline{y}.$$

Moreover, this definition of membership coincides with $\mathtt{E}$ on $\widehat{\mathcal{U}}$.

5.5 Constructing τ via games

Let $x, y \in {}^*\mathcal{U}$ and consider the familiar bisimulation game, say $\mathrm{bisim}(x, y)$, on x, y, constructed as follows. Fix $x_0 = x$ and $y_0 = y$. Right, playing first in the position (x_i, y_i), picks any element $x_{i+1} \in x_i$ or $y_{i+1} \in y_i$, to move to (x_{i+1}, y_i) or (x_i, y_{i+1}) respectively. Left is then required to pick a member of the opposite set (for example, if Right moves to (x_{i+1}, y_i), Left must choose $y_{i+1} \in y_i$) and move to (x_{i+1}, y_{i+1}). From here play continues in $\mathrm{bisim}(x_{i+1}, y_{i+1})$. In general the first player who cannot move is the loser, and an infinite play counts as a draw.

We define a variant of this game, denoted $B(x,y)$, as follows. Player I chooses an element u of x or y, and II responds by selecting an element v of the opposite set (y or x respectively). Now II may respond by choosing a set v' which is "close" to v, and incur a penalty, namely a value infinitesimally close to $\tau(v,v')$. Play then continues in $B(u,v')$.

A link can now be likened to a second-player strategy in $B(x,y)$, and a chain to a composition of such strategies (and so a chain is also similar to a strategy). We denote a link from x to y by $p\colon x \circledcirc y$, and a chain between x and y by $a\colon x \rightsquigarrow y$ (the precise definitions are not symmetric; however whenever $p\colon x \circledcirc y$ or $a\colon x \rightsquigarrow y$ there is an equivalent link or chain in the opposite direction). The weight of a link is then the supremum of all potential penalties for Left when playing by that strategy.

Then $\sigma(x,y)$ is defined by effectively minimising the potential penalty to Left, over all available second-player strategies in the game $B(x,y)$. Contrastingly ρ becomes a measure of how long Left must play in order to keep his penalty negligible (effectively zero, or infinitesimal).

In fact this discussion highlights a third potentially interesting metric. If, say, η, were to assign to (x,y) the cheapest strategy f in $B(x,y)$ such that f survives countably many turns, but also manages to terminate before some fixed point (in time, space or processing power, for example), then we might define

$$\eta(x,y) = \inf\{\mathrm{w}(p)\colon p\ \lfloor p\rfloor > \mathbb{N};\ \alpha\lceil p\rceil < \beta;\ p\colon x \circledcirc y\},$$

where β represents a favourable limit or resources. Unlike ρ, η is not an ultrametric; though it is reasonable to expect that η would behave more favourably than σ, since it limits the number of chains and links under consideration, like ρ.

Of course this interpretation is in some senses different to the construction given in this chapter, mostly due to the fact that ${}^*\mathcal{U}$ is a nonstandard structure, where many of the objects under consideration are infinite but *finite. It would make interesting further research to see how a similar method might be used in general collections of games (specifically, in architectures satisfying some basic set theory, rather than just collections of bisimulation games) for various purposes, such as introducing new quotient spaces; defining pseudometrics and so relevant topological structure; and even for interesting set-theoretical constructions.

CHAPTER 6

CONCLUSIONS

In chapters 2 and 3 we proposed a framework for the study of combinatorial games (under the normal play convention). This includes as examples the majority of games (in particular all the non-loopy, non-misère games) presented in ONAG and Winning Ways. We were able to explain the usefulness of the value map defined there in terms of this framework, as an adjunction (with extra structure). Further, the architectures of chapter 3 continue the studies of Joyal [19] and Cockett et al. [6] in uniting the set-theoretic and strategic aspects of combinatorial games. This framework must now be applied in further research.

A particular area in need of development within this framework is the notion of amphimorphism, in the context of game categories and the additional structure we are likely to assume. In the simpler context of two-ordered groups amphimorphisms seem more attuned to the multiplicative structure than their mono-directional counterparts; therefore we might expect as much with, and are certainly justified in searching for a feasible generalisation to, (probably monoidal) game categories.

The material from chapter 4 is relatively complete: we have defined a theory of amphisets which both encapsulates that of Conway games, and is bi-interpretable with ZF. Some questions, however, have arisen, in particular regarding the foundation axiom. Since there are two separate memberships, many variants of Foundation are possible. For instance, we might choose an axiom, say $\mathrm{wf}_{\in_\mathrm{L}} \wedge \mathrm{wf}_{\in_\mathrm{R}}$, stating that each of $\in_\mathrm{L}, \in_\mathrm{R}$ is wellfounded. In some senses this would constitute a more intuitively correct axiom, and so the following is of some importance.

Conjecture 6.0.1. There exist models of $\mathrm{ZF_2}^-$ (that is, ZF_2 without the axiom of Foundation) which satisfy $\mathrm{wf}_{\in_\mathrm{L}} \wedge \mathrm{wf}_{\in_\mathrm{R}}$, as well as $\neg\,\mathrm{Found}$.

Question 7. Assuming the conjecture to be true, is $\mathrm{ZF}_2 + \bigwedge_\mathrm{P} \mathrm{wf}_{\in_\mathrm{P}} + \neg\,\mathrm{Found}$ synonymous with a subtheory T of ZF? If so, what significance does T have in the context of sets?

In chapter 5 we discussed the construction of various topological set theories on a nonstandard model. These constructions have the potential to be quite fruitful, though we can identify areas where improvement is necessary. We avoided discussion of separation properties within the structures $\widehat{\mathcal{U}}$, and in particular we have not proved that for all $x, y \in \widehat{\mathcal{U}}$ the intersection $x \cap y$ exists. The difficulty arises because of our nonstandard construction: ideally we would show that the intersection $x \cap y$ is the limit of some sequence of sets (for instance, if we took z_n to be, say, the intersection $B_{a_n}(x) \cap B_{a_n}(y)$ for a decreasing sequence (a_n)), and then use a closure property of $\widehat{\mathcal{U}}$ to show that this set is

included. The completeness of $\widehat{\mathcal{U}}$ in particular may be of use here. It is not immediately clear how we would construct such a sequence, however, since the metric itself is not internal. It may be possible to instead circumnavigate this issue by assuming stronger axioms in the original structure $\mathcal{U}$. A further possibility, made feasible by proposition 5.4.2, is to expand to the power set of $\widehat{\mathcal{U}}$ in such a way that the set theory is in some way maintained.

Yet another possibility, which might require reasonable work, is to expand ${}^*\mathcal{U}$ before taking the quotient by $\sim$. This might be possible in a similar manner to that associated with Loeb measures (in fact, introducing a measure which bears some relationship to τ might simplify several other problems, and is not particularly unrealistic). Indeed, if $\mathcal{W} \supseteq {}^*\mathcal{U}$ was closed under countable intersection (if a measure were introduced $\mathcal{W}$ would be a σ-ring, with various advantages), with τ appropriately defined, then $\hat{x} \cap \hat{y}$ could easily be defined as a countable intersection of internal sets. It is not even that difficult to extend our definition of τ on such a class $\mathcal{W}$, since τ already behaves like the Hausdorff distance, by Corollary 5.4.2. There are also obvious definitions extending $\mathtt{E}$; perhaps the most suitable is to set $x \mathrel{\mathtt{E}} y$ if and only if there are sequences $(x_i), (y_i)$ converging to x, y respectively, such that $x_i \in y_i$ for all i. This definition can be seen to coincide with that of $\mathtt{E}$ on internal sets. It is not clear from these definitions whether $\mathcal{W}$ would be closed under various axioms of set theory, such as Pair, Union, etc. If not it would be necessary to extend $\mathcal{W}$ in iterations, which could be problematic. We phrase these curiosities as the following.

Question 8. Is it possible to augment this construction in such a way that the resulting set theory satisfies a reasonable collection of separation, or even replacement, axioms?

Although the potential lack of an intersection is an issue from a set-theoretic perspective, this does not necessarily affect other uses. For instance, if we are interested in $\widehat{\mathcal{U}}$ (or, more likely, a variation of this construction applied to a model of some fragment of ZF_2) as a structure of combinatorial games, then intersection may be unnecessary.

This may not be the only application in the context, however. It could be worthwhile exploring the effects of similar constructions in general on categories of games (specifically on architectures, as defined in chapter 3). There are two options: firstly, to consider such spaces built on top of nonstandard models of hereditarily finite games; and secondly, to adopt our approach to the general case where we have a collection of potentially infinite games.

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